Asymptotic Properties of Branching Symmetric Markov Processes

Yuichi SHIOZAWA

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Contents

Introduction

It is well known that long time behaviors of Markov processes may follow rules such as central limit theorems, laws of large numbers, and large deviation principles. Some rules are controlled by principal eigenvalues. For example, M. Kac [37] proved that for a transient Brownian motion on $\mathbb{R}^d$, the tail probability of the total occupation time on a compact set decays exponentially and its rate is given by the principal eigenvalue of the generator of a time changed Brownian motion. He also proved that the decay rate of a Feynman-Kac semigroup is given by the principal eigenvalue of the associated Schrödinger operator. Nowadays, this fact follows as a corollary of the Donsker-Varadhan large deviation theory ([23]). In this thesis, we study long time asymptotic properties of branching symmetric Markov processes in terms of the principal eigenvalues and the ground states of the associated Schrödinger type operators. In particular, we consider the extinction property, the growth rate of the numbers of particles, and the asymptotic distributions of particles.

A branching symmetric Markov process is known as a simple model of an evolving population. Roughly speaking, a branching symmetric Markov process is described as follows: each particle moves on a state space according to the law of a symmetric Markov process until the splitting time, and then it creates new particles. After that, each of these particles repeats this movement independently. More precisely, let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbf{M} = (X_t, P_x)$ be an m-symmetric Markov process on X and $\overline{\mathbf{M}} = (\mathbf{X}_t, \mathbf{P}_x)$ the branching symmetric Markov process such that each particle moves independently according to the law of $\mathbf{M}$. We denote by μ the *branching rate*, that is, the positive continuous additive functional (PCAF) A_t^{μ} in the Revuz correspondence to μ determines the distribution of the splitting time of each particle. We assume that μ is a *Green-tight* measure (in notation, $\mu \in \mathcal{K}_{\infty}$). For the definition of the Green-tightness, see Definition 1.1. We denote by $\{p_n(x)\}_{n\geq 0}$ the *branching mechanism*, that is, a particle splits into n particles with probability $p_n(x)$ at branching site $x \in X$. Further, let $Q(x) = \sum_{n=1}^{\infty} np_n(x)$ be the expected number of particles which are born at branching site $x \in X$ and define the *intensity of population growth* by $\nu(dx) := (Q(x) - 1)\mu(dx)$.

We first establish a criterion for $\overline{\mathbf{M}}$ to extinct or extinct locally in terms of the principal eigenvalue for a time changed process. We define a formal operator by

$$\check{\mathcal{L}}^{\mu,Q\mu} := \frac{1}{Q\mu}(\mathcal{L} - \mu), \tag{1}$$

where $\mathcal{L}$ is the generator of $\mathbf{M}$. We then see that $\check{\mathcal{L}}^{\mu,Q\mu}$ is regarded as the generator of the $\exp(-A_t^{\mu})$-subprocess time changed with respect to $A_t^{Q\mu}$, where $A_t^{Q\mu}$ is the PCAF corresponding to the measure $Q\mu$. Since $Q\mu$ and μ denote the intensity of creations and the intensity of killings respectively, we say that the operator $\check{\mathcal{L}}^{\mu,Q\mu}$ expresses the balance between these intensities. This suggests that the extinction of the branching process is controlled by the principal eigenvalue of $\check{\mathcal{L}}^{\mu,Q\mu}$. In fact, the operator $\check{\mathcal{L}}^{\mu,Q\mu}$ is realized as a self-adjoint operator on $L^2(X;Q\mu)$ and $\check{\lambda} := \check{\lambda}(Q\mu,\mu)$ denotes the bottom of the spectrum of $\check{\mathcal{L}}^{\mu,Q\mu}$, namely,

$$\check{\lambda}(Q\mu,\mu) = \inf\left\{\mathcal{E}(u,u) + \int_X u^2\,d\mu : u \in \mathcal{F},\ \int_X u^2\,Qd\mu = 1\right\}. \tag{2}$$

Here $(\mathcal{E},\mathcal{F})$ is the Dirichlet form generated by $\mathbf{M}$. We then show that, under the assumption that $\check{\lambda}$ is a discrete spectrum, the branching process $\overline{\mathbf{M}}$ extincts or extincts locally if and only if $\check{\lambda} \geq 1$ (see Theorems 2.4 and 2.11 below).

The extinction problem is one of the basic problems of branching Markov processes and has been studied by many persons. For instance, Sevast'yanov [49] and S. Watanabe [61] considered this problem for a branching Brownian motion on a bounded domain in $\mathbb{R}^d$ with state-independent branching rate and branching mechanism. They then gave a criterion for extinction by the principal eigenvalue of the Dirichlet Laplacian. R. G. Pinsky [42] investigated and developed the theory of generalized principal eigenvalues of Schrödinger operators. Using this theory, he in [43] analytically gave a criterion for a measure-valued branching diffusion process to extinct locally, that is, the particles on every compact set disappear. On the other hand, Engländer and Kyprianou [26] probabilistically gave the criterion for local extinction. In these papers, the ground state of the Schrödinger operator plays an essential role; they construct the ground state by using well-known facts for elliptic differential operators, Harnack's inequality and Schauder's estimate. Here we consider the branching jump Markov processes and this approach is not applicable to construct the ground state because we do not know the corresponding properties for non-local operators. To overcome this difficulty, we use the generator of a time changed process. We can then construct the ground state by using the compact embedding of the Dirichlet form corresponding to the motion component; however, we must restrict the branching rate within the class $\mathcal{K}_\infty$ to show the compact embedding. Further, to prove the regularity of the ground state, we need to restrict the branching rate within the subclass $\mathcal{S}_\infty \subset \mathcal{K}_\infty$, which is introduced in [14] (see Definition 1.1). This assumption on the branching rate essentially says that the branching is rare at infinity. Here we would like to emphasize that our result is an extension of the result in [49] and [61] because every constant function belongs to $\mathcal{S}_\infty$ for Brownian motions on bounded domains. Moreover, we allow the state spaces to be unbounded and the branching rate to be not only functions but also measures.

We next study the exponential growth of the numbers of particles for the branching process $\overline{\mathbf{M}}$. To cope with this problem, we use the principal eigenvalue and the ground state of an associated Schrödinger operator. More precisely, let

$$\mathcal{L}^\nu := \mathcal{L} + \nu \tag{3}$$

and denote by $\lambda_1 := \lambda_1(\nu)$ the bottom of the spectrum of $\mathcal{L}^\nu$:

$$\lambda_1(\nu) = \inf\left\{\mathcal{E}(u,u) - \int_X u^2\,d\nu : u \in \mathcal{F}, \int_X u^2\,dm = 1\right\}. \tag{4}$$

Let h be the corresponding ground state. Namely, h is a function on X attaining the infimum of (4). Define

$$M_t = e^{\lambda_1 t}\sum_{i=1}^{Z_t} h\left(\mathbf{X}_t^i\right), \quad t \geq 0,$$

where Z_t denotes the total number of particles and $\mathbf{X}_t^i$, $1 \leq i \leq Z_t$, is the position of the ith particle at time t. Then, under the assumption that λ_1 is a negative discrete spectrum, we prove the square integrability of the martingale M_t. As a result, a limit $M_\infty := \lim_{t\to\infty} M_t$ exists in $L^1(\mathbf{P}_x)$ and $\mathbf{P}_x$-a.s. Furthermore, we show that the limit M_∞ is positive $\mathbf{P}_x$-a.s. on the event that the branching process survives (Theorem 3.4). This result says that Z_t grows exponentially with rate $-\lambda_1$ at least. We also study the exponential growth of the number of particles in every relatively compact open set (Theorem 3.8). Theorem 3.8 indicates that the number of particles in every relatively compact open set may grow exponentially at rate $-\lambda_1$. Engländer and Kyprianou [26] studied the same problem for a branching diffusion process with regular branching rate function. Here we consider more general branching symmetric Markov processes than those studied in [26]. Indeed, we discuss the exponential growth for the branching processes whose motion components are jump Markov processes and whose branching rates are measures.

As stated above, the square integrability of M_t is crucial. We now explain how to prove it. By the definition of the branching symmetric Markov process, it follows that

$$\begin{aligned}\mathbf{E}_x\left[M_t^2\right] &= e^{2\lambda_1 t} E_x\left[\exp\left(A_t^\nu\right) h(X_t)^2; t<\zeta\right] \\ &\quad + E_x\left[\int_0^{t\wedge\zeta} \exp\left(2\lambda_1 s + A_s^\nu\right) h(X_s)^2\, dA_s^{R\mu}\right],\end{aligned} \tag{5}$$

where ζ is the lifetime of $\mathbf{M}$, $A_t^\nu = A_t^{Q\mu} - A_t^\mu$ and $R(x) = \sum_{n=0}^\infty n(n-1)p_n(x)$. Hence, to show the square integrability of M_t, we use a criterion for the gaugeability of measures. Here $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$ is said to be *gaugeable* if

$$\sup_{x\in X} E_x\left[\exp\left(A_\zeta^\mu\right)\right] < \infty.$$

Z.-Q. Chen [14] and Takeda [52] then showed that μ is gaugeable if and only if $\check{\lambda}(\mu^+,\mu^-) > 1$ (see Theorem 1.2 below). In addition, there are relations between $\lambda_1(\mu)$ and $\check{\lambda}(\mu^+,\mu^-)$ as follows:

$$\lambda_1(\mu) \geq 0 \iff \check{\lambda}(\mu^+,\mu^-) \geq 1 \quad \text{and} \quad \lambda_1(\mu) > 0 \Longrightarrow \check{\lambda}(\mu^+,\mu^-) > 1.$$

Applying these results to the right hand side of (5), we establish the square integrability of M_t.

We finally establish limit theorems for a class of branching symmetric Markov processes. Namely, under the assumption that λ_1 is a negative discrete spectrum, we show that for any $x \in X$, $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(A) = M_\infty \int_A h\, dm \tag{6}$$

for every relatively compact Borel set A in X, where $Z_t(A)$ denotes the number of particles on the set A at time t (Theorem 4.7). The equation (6) says that $Z_t(A)$ grows exponentially at rate $-\lambda_1$ and that the ground state determines the asymptotic distribution of particles. The limit theorem for branching symmetric Markov processes has been studied for a long time. For example, S. Watanabe studied in [61] and [62] the asymptotic properties of branching symmetric diffusion processes and established a limit theorem in [62]. His approach is based on a generalization of the Fourier transform and requires that the transition densities of the Feynman-Kac semigroups are represented by the spectral measures and the eigenfunctions. Asmussen and Hering [4] also established a limit theorem in [4] for general supercritical branching processes. To apply their result to branching symmetric Markov processes, we have to check that every spectrum of the Schrödinger operator is discrete, and consequently the Feynman-Kac semigroup has an eigenfunction expansion. However, branching symmetric α-stable processes with singular branching rates do not satisfy the conditions imposed in [62] and [4]. In fact, since the transition density of $\mathcal{L}^\nu$ may not be expressed by the spectral measure, the methods used in S. Watanabe [62] and in Asmussen and Hering [4] are not applicable here. Unlike their conditions, we use the fact that the operator $\mathcal{L}^\nu$ has a spectral gap. A crucial point is that the spectral gap implies the ergodicity of the h-transformed semigroup of the Feynman-Kac semigroup. By this property with an application of the gaugeability of measures, we can establish (6) in discrete time, and then extend it to a continuous time version by applying a method from the proof of Theorem 1' in [4].

We consider branching Brownian motions and branching symmetric α-stable processes as concrete models; a Brownian motion is a typical model of diffusion processes and a symmetric α-stable process is a typical model of jump processes. As we saw above, we need to calculate explicitly the principal eigenvalues $\check{\lambda}$ and λ_1 in order to find asymptotic properties for these processes. However, it is difficult in general to calculate the principal eigenvalues of Schrödinger type operators with non-local principal parts. Therefore, for special classes of them, we calculate the principal eigenvalues by the Dirichlet principle. We can then obtain asymptotic properties explicitly for a class of branching Brownian motions and branching symmetric α-stable processes. For example, let us consider a branching symmetric α-stable process in one dimension with $1 < \alpha \leq 2$. First take the Dirac measure at $a > 0$ as branching rate and suppose that each particle dies upon arriving at 0. We then see that this branching symmetric α-stable process extincts if and only if

$$0 < a \leq \left\{ -\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{2} \right\}^{1/(\alpha-1)}$$

(Example 2.18). Next take δ_0, the Dirac measure at the origin, as branching rate and suppose that the state space is $\mathbb{R}$. We then obtain for any $x \in \mathbb{R}$, $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{\lambda_1(\alpha)t} Z_t((-r,r)) = \begin{cases} \left(\int_{-\infty}^{\infty} h\,dx - O\left(r^{-\alpha}\right)\right) M_\infty, & 1 < \alpha < 2 \\ 2(1-e^{-r})\, M_\infty, & \alpha = 2 \end{cases}$$

for any $r > 0$, where

$$\lambda_1(\alpha) = -\left\{ \frac{2^{1/\alpha}}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \right\}^{\alpha/(\alpha-1)}$$

is the principal eigenvalue of the Schrödinger operator $-\frac{1}{2}\left(-\Delta\right)^{\alpha/2} + \delta_0$ and h is the corresponding ground state. Moreover, M_∞ is positive $\mathbf{P}_x$-a.s. (Example 4.12).

Since the explicit calculations of the principal eigenvalues are difficult as we mentioned above, we try to give lower bound estimates. To do this, we first establish a variational formula for Dirichlet forms. Recall that X be a locally compact separable metric space and m is a positive Radon measure on X with full support. In [24], Donsker and Varadhan proved a large deviation principle of occupation distributions of conservative Markov processes on X with the so-called I-function as its rate function. Moreover, they showed that, if the Markov process is m-symmetric, then the I-function is identified with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$. M. F. Chen [13] then extended this identification to symmetric jump processes with killings. Our objective is to extend it further to general symmetric Markov processes including time changed processes. More precisely, let $\hat{\mathcal{L}}$ be the "extended generator" of a symmetric Markov process determined by the martingale problem and $\mathcal{D}^+\left(\hat{\mathcal{L}}\right)$ the set of nonnegative functions in the domain of $\hat{\mathcal{L}}$ (see Definition 5.1). We then prove

$$\mathcal{E}(f,f) = -\inf_{u\in\mathcal{D}^+(\hat{\mathcal{L}}),\,\varepsilon>0}\int_X \frac{\hat{\mathcal{L}}u}{u+\varepsilon}f^2dm, \quad f\in\mathcal{F}. \tag{7}$$

Furthermore, applying this formula, we obtain the lower bound estimate of the bottom of the spectrum: let λ_0 be the bottom of the spectrum of the operator $\mathcal{L}$. We can then derive the following *generalized Barta's inequality*,

$$\lambda_0 \geq \inf_{x\in X}\left(\frac{-\hat{\mathcal{L}}u}{u}\right)(x), \quad u\in\mathcal{D}^{++}\left(\hat{\mathcal{L}}\right),$$

where $\mathcal{D}^{++}\left(\hat{\mathcal{L}}\right)$ is the set of strictly positive functions in the domain of $\hat{\mathcal{L}}$ (Theorem 5.8).

The organization of this thesis is as follows. In Chapter 1, we first recall the notions of Dirichlet forms and symmetric Markov processes. We next introduce two classes $\mathcal{K}_\infty$ and $\mathcal{S}_\infty$ of Kato measures, which play an important role in this thesis. We next introduce the notion of branching symmetric Markov processes. We finally introduce the notion of symmetric α-stable processes because we consider branching symmetric α-stable processes as typical models. Chapters 2, 3 and 4 are devoted to the study of asymptotic properties of branching symmetric Markov processes and its applications to branching Brownian motions and branching symmetric α-stable processes. We give in Chapter 2 a criterion for extinction or local extinction in terms of the principal eigenvalues for time changed processes. We study in Chapters 3 and 4 the exponential growth of the numbers of particles and the asymptotic distributions of particles in terms of the principal eigenvalues and the ground states of Schrödinger operators. Chapters 5 and 6 are devoted to calculations and estimates of the principal eigenvalues for Schrödinger type operators. We establish in Chapter 5 a variational formula for Dirichlet forms generated by general symmetric Markov processes. As its application, we derive generalized Barta's inequality. Using this inequality and the Dirichlet principle, we estimate and calculate in Chapter 6 the principal eigenvalues of Schrödinger type operators associated with Brownian motions and symmetric α-stable processes. In Appendix A, we show that the Green function is positive for any absorbing symmetric α-stable process on an open set. This result implies that any absorbing α-stable process is irreducible even if the state space is disconnected.

Chapter 4 is based on a joint work with Zhen-Qing Chen and Chapter 5 is based on a joint work with Masayoshi Takeda.

Chapter 1

Preliminaries

In this chapter, we first review the general theory of Dirichlet forms and some facts related to Feynman-Kac functionals. We next introduce the notion of branching Markov processes. We finally introduce the notion of symmetric α-stable processes and remark some properties.

1.1 Dirichlet forms and symmetric Hunt processes

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbf{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P_x, \zeta)$ be an m-symmetric Hunt process on X, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimal admissible filtration. The shift operator θ_t satisfies $X_t \circ \theta_s = X_{t+s}$ identically for $s,\ t \geq 0$, and ζ is the lifetime, $\zeta = \inf\{t > 0 : X_t = \Delta\}$, where Δ is the cemetery point.

Let us denote by$(\mathcal{E}, \mathcal{F})$ the regular Dirichlet form of $\mathbf{M}$. Let $\mathcal{F}_e$ be the family of m-measurable functions on X such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. Then $(\mathcal{E}, \mathcal{F}_e)$ is called the extended Dirichlet form of $(\mathcal{E}, \mathcal{F})$ ([29, p. 36]). We define the 1-capacity associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ for an open set $O \subset X$ by

$$\mathrm{Cap}_{(1)}(O) = \inf\left\{\mathcal{E}_1(u,u) : u \in \mathcal{F},\ u \geq 1 \ m\text{-a.e. on } O\right\}, \tag{1.1}$$

where $\mathcal{E}_\alpha(u,u) = \mathcal{E}(u,u) + \alpha \int_X u^2\, dm$ for $\alpha > 0$, and for any set $A \subset X$ by

$$\mathrm{Cap}_{(1)}(A) = \inf\left\{\mathrm{Cap}_{(1)}(O) : O \text{ is open},\ O \supset A\right\}.$$

For a set $A \subset X$, a statement depending on $x \in A$ is said to hold q.e. on A, if there exists a set $N \subset A$ of zero capacity such that the statement is true for $x \in A \setminus N$. Here q.e. is an abbreviation for *quasi everywhere.* A function $u \in \mathcal{F}_e$ is said to be *quasi continuous*, if for any $\varepsilon > 0$, there exists an open set $O \subset X$ with $\mathrm{Cap}_{(1)}(O) < \varepsilon$ such that $u|_{X\setminus O}$ is finite continuous, where $u|_{X\setminus O}$ is the restriction of u on $X \setminus O$. It is then known in Theorem 2.1.7 of [29] that each $u \in \mathcal{F}_e$ admits a quasi continuous m-version. In the sequel, we always assume that each $u \in \mathcal{F}_e$ is quasi continuous.

An increasing sequence $\{F_n\}$ of closed sets is said to be a *nest* if $\lim_{n\to\infty} \mathrm{Cap}_{(1)}(X \setminus F_n) = 0$. An increasing sequence $\{F_n\}$ of closed sets is said to be a *generalized nest* if $\lim_{n\to\infty} \mathrm{Cap}_{(1)}(K \setminus F_n) = 0$ for any compact set $K \subset X$. A positive Borel measure μ is said to be *smooth*, if μ charges no set of zero capacity and there exists a generalized nest $\{F_n\}$ such that $\mu(F_n) < \infty$

for all n. Denote by $\mathcal{S}$ the set of smooth measures. It is then known in Theorem 5.1.4 of [29] that there exists a one to one correspondence between smooth measures and positive continuous additive functionals (PCAFs in abbreviation), the so-called *Revuz correspondence*, as follows; if we denote by A_t^μ the PCAF corresponding to $\mu \in \mathcal{S}$, then for any γ-excessive function h ($\gamma \geq 0$) and any positive Borel measurable function f,

$$\lim_{t \downarrow 0} \frac{1}{t} \int_X E_x \left[\int_0^t f(X_s)\, dA_s^\mu \right] h(x)\, m(dx) = \int_X f(x)h(x)\, \mu(dx).$$

A positive Radon measure μ on X is said to be of *finite energy integral*, if

$$\int_X |u|\, d\mu \leq C\sqrt{\mathcal{E}_1(u,u)}, \quad u \in \mathcal{F} \cap C_0(X) \tag{1.2}$$

for some positive constant C, where $C_0(X)$ stands for the set of continuous functions on X with compact support. Denote by $\mathcal{S}_0$ the set of measures of finite energy integral. Then, by the Riesz representation theorem, there exists a unique function $G_\alpha\mu \in \mathcal{F}$ for $\mu \in \mathcal{S}_0$ such that

$$\mathcal{E}_\alpha(G_\alpha\mu, u) = \int_X u\, d\mu, \quad u \in \mathcal{F} \tag{1.3}$$

for any $\alpha > 0$ ([29, Theorem 2.2.5]). We call $G_\alpha\mu$ the *α-potential of μ*.

For any $\mu \in \mathcal{S}$, there exists a generalized nest $\{F_n\}$ such that $\mu_n = \mathbf{1}_{F_n} \cdot \mu \in \mathcal{S}_0$ ([29, Theorem 2.2.4]). Then Lemma 2.2.10 of [29] implies that $G_\alpha\mu_n - G_\alpha\mu_m$ is again an α-potential for $n > m$, and consequently $\{G_\alpha\mu_n\}$ is an increasing sequence. We thus define the α-potential of $\mu \in \mathcal{S}$ by $G_\alpha\mu = \lim_{n\to\infty} G_\alpha\mu_n$. We next characterize $G_\alpha\mu$ probabilistically. Let A_t^μ be the PCAF whose Revuz measure is μ. Then

$$G_\alpha\mu_n(x) = E_x \left[\int_0^\zeta e^{-\alpha t} \chi_{F_n}(X_t)\, dA_t^\mu \right] \quad \text{q.e. } x \in X.$$

Since $\{F_n\}$ is a generalized nest, it holds that

$$\lim_{n\to\infty} E_x \left[\int_0^\zeta e^{-\alpha t} \chi_{F_n}(X_t)\, dA_t^\mu \right] = E_x \left[\int_0^\zeta e^{-\alpha t}\, dA_t^\mu \right].$$

Hence we get

$$G_\alpha\mu(x) = E_x \left[\int_0^\zeta e^{-\alpha t}\, dA_t^\mu \right] \quad \text{q.e. } x \in X. \tag{1.4}$$

Denote by $\mathcal{S}_1$ the set of positive Radon measures on X charging no set of zero capacity. Then $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}$.

When $\mathbf{M}$ is transient, the 0-order capacity $\mathrm{Cap}(A)$ is defined by replacing $\mathcal{E}_1$ and $\mathcal{F}$ in (1.1) with $\mathcal{E}$ and $\mathcal{F}_e$, respectively. We say that a positive Radon measure μ on X is said to be *of finite* (*0-order*) *energy integral*, if the inequality (1.2) holds with $\mathcal{E}_1$ on the right hand side replaced by $\mathcal{E}$. Denote by $\mathcal{S}_0^{(0)}$ the set of measures of 0-order finite energy integral. Then the equation (1.3) with $\alpha = 0$ determines a unique function $G\mu \in \mathcal{F}_e$ for any $\mu \in \mathcal{S}_0^{(0)}$. We call $G\mu$ the (*0-order*) *potential* of μ. By the same argument as above, we can define $G\mu$ for any $\mu \in \mathcal{S}$ by $G\mu = \lim_{n\to\infty} G\mu_n$, where $\mu_n = \mathbf{1}_{F_n} \cdot \mu$ and F_n is a generalized nest such that $\mu_n \in \mathcal{S}_0$. We also see that

$$G\mu(x) = E_x \left[A_\zeta^\mu \right] \quad \text{q.e. } x \in X. \tag{1.5}$$

Let (N, H) be a *Lévy system* of $\mathbf{M}$ (see [8] and [29, Theorem 5.3.1]); that is, N is a kernel on $(X_\Delta, \mathcal{B}(X_\Delta))$ such that $N(x, \{x\}) = 0$ for any $x \in X$ and H_t is a PCAF of $\mathbf{M}$ such that, for any nonnegative function $\phi \in \mathcal{B}(X_\Delta \times X_\Delta)$ with $\phi(x, x) = 0$ for any $x \in X_\Delta$,

$$E_x\left[\sum_{s \le t} \phi(X_{s-}, X_s)\right] = E_x\left[\int_0^t \int_{X_\Delta} \phi(X_s, y) N(X_s, y)\, dH_s\right],$$

where $X_\Delta = X \cup \{\Delta\}$ and $X_{t-} = \lim_{s \uparrow t} X_s$. Denote by μ_H the Revuz measure of the PCAF H_t of $\mathbf{M}$ and define

$$J(dx, dy) := N(x, dy)\mu_H(dx) \qquad \text{and} \qquad \kappa(dx) := N(x, \Delta)\mu_H(dx), \tag{1.6}$$

which are called the jump measure and the killing measure of $\mathbf{M}$, respectively.

By Fukushima's decomposition [29, Theorem 5.2.2], it holds that for $u \in \mathcal{F}_e$,

$$u(X_t) - u(X_0) = M_t^u + N_t^u, \quad t \ge 0 \quad P_x\text{-a.s. for q.e. } x \in X, \tag{1.7}$$

where M_t^u is a martingale additive functional of finite energy and N_t^u is a continuous additive functional of zero energy. Denote by $M_t^{u,c}$ and $\mu_{\langle M^{u,c}\rangle}$, respectively, the continuous martingale part of M_t^u and the Revuz measure corresponding to $\langle M^{u,c}\rangle_t$, the quadratic variation of $M_t^{u,c}$. The measure $\mu_{\langle M^{u,c}\rangle}$ is called the *energy measure* of $M_t^{u,c}$. A Beurling-Deny decomposition ([29, Theorem 5.3.1]) then implies that

$$\mathcal{E}(u, u) = \frac{1}{2}\int_X \mu_{\langle M^{u,c}\rangle}(dx) + \frac{1}{2}\iint_{X \times X \setminus \triangle} (u(x) - u(y))^2\, J(dx, dy) + \int_X u(x)^2\, \kappa(dx), \quad u \in \mathcal{F},$$

where $\triangle = \{(x, y) \in X \times X : x = y\}$.

Let $\mathbf{M}^\mu = (X_t^\mu, P_x^\mu)$, $\mu \in \mathcal{S}_1$, be the subprocess of $\mathbf{M}$ with respect to the multiplicative functional $\exp(-A_t^\mu)$ (see [29, Appendix A.2] for details):

$$E_x^\mu\left[f(X_t^\mu)\right] = E_x\left[\exp(-A_t^\mu) f(X_t); t < \zeta\right].$$

Then $\mathbf{M}^\mu$ generates the regular Dirichlet form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ ([29, Theorem 6.1.1]):

$$\begin{aligned} \mathcal{F}^\mu &= \mathcal{F} \cap L^2(X; \mu) \\ \mathcal{E}^\mu(u, u) &= \mathcal{E}(u, u) + \int_X u^2\, d\mu, \quad u \in \mathcal{F}^\mu. \end{aligned}$$

Denote by τ_t^μ the right continuous inverse of A_t^μ,

$$\tau_t^\mu = \inf\left\{s > 0 : A_{s \wedge \zeta}^\mu > t\right\}.$$

Let $F = \operatorname{supp}[\mu]$ and let F^μ be the *fine support* of the measure μ defined by

$$F^\mu = \{x \in X : P_x(\tau_0^\mu = 0) = 1\}. \tag{1.8}$$

Note that F^μ is finely closed and $A_t^\mu(\omega)$ increases only when $X_t(\omega) \in F^\mu$ for P_x-a.s. $\omega \in \Omega$ for q.e. $x \in X$ ([29, Lemma 5.1.11]). The *time changed process* $\check{\mathbf{M}} = (Y_t^\mu, P_x)$ of $\mathbf{M}$ with respect to A_t^μ is defined by $Y_t^\mu = X_{\tau_t^\mu}$. Then $\check{\mathbf{M}}$ is a μ-symmetric Hunt process on F^μ with lifetime A_ζ^μ ([29, Theorem 6.2.1]). Set

$$H_{F^\mu} u(x) = E_x \left[u(X_{\sigma_{F^\mu}}); \sigma_{F^\mu} < \infty \right],$$

where σ_{F^μ} is the hitting time of F^μ, $\sigma_{F^\mu} = \inf\{t > 0 : X_t \in F^\mu\}$. Then $\check{\mathbf{M}}$ generates the regular Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(F;\mu)$ ([29, Theorem 6.2.1]):

$$\begin{aligned} \check{\mathcal{F}} &= \left\{ \psi \in L^2(F;\mu) : \psi = u \ \ \mu\text{-a.e. on } F \text{ for some } u \in \mathcal{F}_e \right\} \\ \check{\mathcal{E}}(\psi,\psi) &= \mathcal{E}(H_{F^\mu} u, H_{F^\mu} u), \quad \psi \in \check{\mathcal{F}}, \ \psi = u \ \ \mu\text{-a.e. on } F \text{ for some } u \in \mathcal{F}_e. \end{aligned}$$

Moreover, $(\check{\mathcal{E}}, \check{\mathcal{F}})$ satisfies

$$\check{\mathcal{E}}(u,u) = \inf\{\mathcal{E}(v,v) : v \in \mathcal{F}_e, \ v = u \ \text{ q.e. on } F\}. \tag{1.9}$$

The equation (1.9) is the so-called *Dirichlet principle.*

1.2 Gaugeability and Feynman-Kac semigroups

1.2.1 Gaugeability and classes of measures

Let $\{p_t, t \geq 0\}$ be the Markovian transition semigroup of $\mathbf{M}$ given by

$$p_t f(x) = E_x \left[f(X_t) \right], \quad f \in \mathcal{B}^+(X),$$

where $\mathcal{B}^+(X)$ denotes the set of nonnegative Borel measurable functions on X. In this subsection, we assume that the transition density of $\mathbf{M}$ is absolutely continuous with respect to m and denote by $p_t(x,y)$ the integral kernel of p_t,

$$p_t f(x) = \int_X p_t(x,y) f(y)\, m(dy).$$

Let $G_\alpha(x,y)$ be the α-resolvent density of $\mathbf{M}$,

$$G_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y)\, dt, \quad \alpha > 0.$$

If $\mathbf{M}$ is transient, then the Green function

$$G_0(x,y) := \int_0^\infty p_t(x,y)\, dt$$

exists for $x \neq y$, and we put $G(x,y) = G_0(x,y)$.

We now introduce classes of measures in $\mathcal{S}$.

Definition 1.1. (i) *A positive smooth Radon measure on X is said to be in the Kato class $\mathcal{K}$, if*

$$\lim_{\alpha \to \infty} \sup_{x \in X} \int_X G_\alpha(x,y)\, \mu(dy) = 0.$$

(ii) *A positive smooth Radon measure on X is said to be in $\mathcal{K}_\infty(G_\alpha)$, if for any $\varepsilon > 0$, there exists a compact set $K \subset X$ and a positive constant $\delta > 0$ such that*

$$\sup_{x \in X} \int_{X \setminus K} G_\alpha(x,y)\, \mu(dy) < \varepsilon,$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{x \in X} \int_B G_\alpha(x,y)\,\mu(dy) < \varepsilon.$$

The class $\mathcal{K}_\infty$ is defined by

$$\mathcal{K}_\infty = \begin{cases} \mathcal{K}_\infty(G), & \mathbf{M} \textit{ is transient} \\ \mathcal{K}_\infty(G_1), & \mathbf{M} \textit{ is recurrent.} \end{cases}$$

(iii) *A positive smooth Radon measure μ on X is said to be in $\mathcal{S}_\infty(G_\alpha)$, if for any $\varepsilon > 0$, there exists a compact set $K \subset X$ and a positive constant $\delta > 0$ such that*

$$\sup_{(x,z) \in X \times X \setminus \triangle} \int_{X \setminus K} \frac{G_\alpha(x,y) G_\alpha(y,z)}{G_\alpha(x,z)}\,\mu(dy) < \varepsilon,$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x,z) \in X \times X \setminus \triangle} \int_B \frac{G_\alpha(x,y) G_\alpha(y,z)}{G_\alpha(x,z)}\,\mu(dy) < \varepsilon.$$

When $\mathbf{M}$ is transient, the class $\mathcal{S}_\infty(G)$ is simply denoted by $\mathcal{S}_\infty$

If $\mathbf{M}$ is transient, then it holds that $\mathcal{S}_\infty \subset \mathcal{K}_\infty$ by Corollary 3.1 of [19] and any measure in $\mathcal{K}_\infty$ with compact support belongs to $\mathcal{S}_\infty$. It is known in Proposition 2.2 of [14] that any measure μ in $\mathcal{K}_\infty$ is *Green bounded*,

$$\sup_{x \in X} E_x\left[A^\mu_\zeta\right] = \sup_{x \in X} \int_X G(x,y)\,\mu(dy) < \infty. \tag{1.10}$$

In the sequel, we assume that $\mathbf{M}$ is transient. Let μ be a signed measure on X which can be decomposed into $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_\infty$. Then the measure μ is said to be *gaugeable*, if

$$\sup_{x \in X} E_x\left[\exp\left(A^\mu_\zeta\right)\right] < \infty,$$

where $A^\mu_t = A^{\mu^+}_t - A^{\mu^-}_t$. Define

$$\check{\lambda}(\mu^+, \mu^-) = \inf\left\{\mathcal{E}(u,u) + \int_X u^2\,d\mu^- : u \in \mathcal{F},\ \int_X u^2\,d\mu^+ = 1\right\}. \tag{1.11}$$

When $\mu^- = 0$, we simply denote $\check{\lambda}(\mu^+, 0)$ by $\check{\lambda}(\mu)$.

Theorem 1.2. ([14, Corollary 2.9, Theorem 5.1]) *Suppose that a signed measure μ on X can be decomposed into $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_\infty$. Then the following conditions are equivalent:*
(i) *The measure μ is gaugeable;*
(ii) $\check{\lambda}(\mu^+, \mu^-) > 1$*;*
(iii) $\sup_{x \in X} E_x\left[\int_0^\zeta \exp\left(A^\mu_t\right)\,dA^\nu_t\right] < \infty$ *for any $\nu \in \mathcal{K}_\infty$.*

The implications (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (ii) are already proved in [14, Corollary 2.9, Theorem 5.1]. We can show the implication (ii) $\Rightarrow$ (iii) in a similar way to that yielding Proposition 3.2 of [15] as follows. Let μ be a measure on X which can be decomposed into $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_\infty$. Assume that $\check{\lambda}(\mu^+, \mu^-) > 1$ and fix a measure $\nu \in \mathcal{K}_\infty$. Since

$$\check{\lambda}(p\mu^+, p\mu^-) \geq \check{\lambda}(p\mu^+, \mu^-) = \frac{1}{p}\check{\lambda}(\mu^+, \mu^-)$$

for any $p > 1$, we can take $p > 1$ such that $\check{\lambda}(p\mu^+, p\mu^-) > 1$. Let $q > 1$ be the conjugate component of p, that is, q satisfies $1/p + 1/q = 1$. Then the Hölder inequality implies that

$$\begin{aligned} E_x\left[\int_0^\zeta \exp\left(A_t^\mu\right) dA_t^\nu\right] &\leq E_x\left[\sup_{0\leq t\leq \zeta} \left(\exp\left(A_t^\mu\right)\right) A_\zeta^\nu\right] \\ &\leq E_x\left[\sup_{0\leq t\leq \zeta} \left(\exp\left(A_t^{p\mu}\right)\right)\right]^{1/p} E_x\left[A_\zeta^{q\nu}\right]^{1/q}. \end{aligned} \tag{1.12}$$

Noting that the measure $q\nu$ belongs to $\mathcal{K}_\infty$, we have $\sup_{x\in X} E_x\left[A_\zeta^{q\nu}\right] < \infty$. A direct calculation yields that

$$\begin{aligned} \sup_{0\leq t\leq \zeta} \left(\exp\left(A_t^{p\mu}\right)\right) &= \sup_{0\leq t\leq \zeta} \int_0^t \exp\left(A_s^{p\mu}\right) dA_s^{p\mu} + 1 \\ &\leq \sup_{0\leq t\leq \zeta} \int_0^t \exp\left(A_s^{p\mu}\right) dA_s^{p\tilde{\mu}^+} + 1 \\ &= \int_0^\zeta \exp\left(A_s^{p\mu}\right) dA_s^{p\tilde{\mu}^+} + 1, \end{aligned}$$

where $\tilde{\mu}^+ - \tilde{\mu}^-$ is the Jordan decomposition of the measure μ. Since the measures $\tilde{\mu}^+$ and $\tilde{\mu}^-$ belong to the class $\mathcal{K}_\infty$, respectively, and the condition that $\check{\lambda}(p\mu^+, p\mu^-) > 1$ is equivalent to that $\check{\lambda}(p\tilde{\mu}^+, p\tilde{\mu}^-) > 1$ by [58, Lemma 3.1], we obtain

$$\sup_{x\in X} E_x\left[\int_0^\zeta \exp\left(A_t^{p\mu}\right) dA_t^{p\tilde{\mu}^+}\right] < \infty$$

by [14, Corollary 2.9, Theorem 5.1]. Therefore, the right hand side of (1.12) is bounded, which shows the implication (ii) $\Rightarrow$ (iii).

1.2.2 Feynman-Kac semigroups

In this subsection, we assume the following on $\mathbf{M}$:

Assumption 1.3. (i) (Irreducibility) If a Borel set A is p_t-invariant, that is, if $p_t(\mathbf{1}_A f) = \mathbf{1}_A p_t f$ holds for every $f \in L^2(X; m) \cap \mathcal{B}_b(X)$ and $t > 0$, then either $m(A) = 0$ or $m(X \setminus A) = 0$ holds. Here $\mathcal{B}_b(X)$ stands for the set of bounded Borel measurable functions on $\mathbf{M}$.
(ii) (Strong Feller property) For any $f \in \mathcal{B}_b(X)$, $p_t f$ is a bounded and continuous function on X.
(iii) (Ultracontractivity) For any $t > 0$, it holds that $\|p_t\|_{1,\infty} < \infty$, where $\|\cdot\|_{p,q}$ denotes the operator norm from $L^p(X; m)$ to $L^q(X; m)$.

Note that, by Assumption 1.3 (ii) and the m-symmetry of p_t, the transition probability of $\mathbf{M}$ is absolutely continuous with respect to m.

We know from [50] that, for a positive smooth measure μ of $\mathbf{M}$ on X and $\alpha > 0$,

$$\int_X u^2 d\mu \le \|G_\alpha \mu\|_\infty \mathcal{E}_\alpha(u,u), \quad u \in \mathcal{F}.$$

Then, by the definition of $\mathcal{K}$, it follows that for $\mu \in \mathcal{K}$, there exists a constant $\alpha > 0$ such that

$$\int_X u^2 d\mu \le \frac{1}{2}\mathcal{E}(u,u) + \alpha \int_X u^2\, dm \qquad \text{for } u \in \mathcal{F}. \tag{1.13}$$

Let μ be a signed measure on E which can be decomposed as $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}$. Let $\{p_t^\mu, t \ge 0\}$ be the Feynman-Kac semigroup given by

$$p_t^\mu f(x) = E_x\left[\exp\left(A_t^\mu\right) f(X_t)\right], \quad f \in \mathcal{B}^+(X). \tag{1.14}$$

Then it follows from [1, Theorem 3.3] and (1.13) above that $\{p_t^\mu, t \ge 0\}$ is a strongly continuous semigroup on $L^2(X;m)$ and its associated quadratic form is $(\mathcal{E}^\mu, \mathcal{F})$ where

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_X u^2\, d\mu, \quad u \in \mathcal{F}.$$

Moreover, under Assumption 1.3, we have from [1] the following.

Theorem 1.4. *Let $\mu = \mu^+ - \mu^- \in \mathcal{K} - \mathcal{K}$. Then, under Assumption 1.3, it holds that*
(i) *For any $f \in \mathcal{B}_b(X)$, $p_t^\mu f$ is a bounded and continuous function on X. Moreover, p_t^μ admits an integral kernel $p_t^\mu(x,y)$ that is jointly continuous in $(x,y) \in X \times X$ for each $t > 0$:*

$$p_t^\mu f(x) = \int_X p_t^\mu(x,y) f(y) m(dy), \quad f \in \mathcal{B}^+(X).$$

(ii) *For any $t > 0$, it holds that $\|p_t^\mu\|_{p,q} < \infty$ for any $1 \le p \le q \le \infty$.*

For a signed measure $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$, define

$$\lambda_1(\mu) = \inf\left\{\mathcal{E}^\mu(u,u) :\ u \in \mathcal{F},\ \ \int_X u^2\, dm = 1\right\}. \tag{1.15}$$

Denote by$\sigma(\mathcal{E}^\mu)$ the totality of the spectrum of the self-adjoint operator associated with $(\mathcal{E}^\mu, \mathcal{F})$. Let

$$\lambda_0 := \inf\left\{\mathcal{E}(u,u) :\ u \in \mathcal{F},\ \ \int_X u^2\, dm = 1\right\}.$$

We also make the following assumption:

Assumption 1.5. (Compact embedding) *The embedding of $(\mathcal{F}, \mathcal{E}_1)$ into $L^2(X;\mu^+)$ is compact, where $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \int_X u^2\, dm$.*

Under this assumption, by the Friedrichs theorem ([40, Lemma 2.5.4/1]), the spectrum of $\sigma(\mathcal{E}^\mu)$ less than λ_0 consists of only isolated eigenvalues with finite multiplicities. We denote by h the corresponding ground state normalized as $\int_X h^2\, dm = 1$. Let $\lambda_2(\mu)$ denote the second bottom of the spectrum of $\sigma(\mathcal{E}^\mu)$, that is,

$$\lambda_2(\mu) = \inf\left\{\mathcal{E}^\mu(u,u) : u \in \mathcal{F},\ \int_X u^2\, dm = 1,\ \int_X uh\, dm = 0\right\}.$$

Then $\lambda_2(\mu) - \lambda_1(\mu) > 0$ if $\lambda_1(\mu) < \lambda_0$.

In the remainder of this section, we fix a signed measure $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Assume that Assumption 1.5 holds and that $\lambda_1 := \lambda_1(\mu) < 0$. We note that, since it holds that $h = e^{\lambda_1 t} p_t^\mu h$ on X, the ground state h is bounded and continuous by Theorem 1.4 and strictly positive by the irreducibility of $\mathbf{M}$ and the strict positivity of $\exp(A_t^\mu)$. Let $G_\alpha^{\mu^-}$ and $G_\alpha^{\mu^-}(x,y)$ denote the α-resolvent and the α-resolvent density respectively, of the $\exp(-A_t^\mu)$-subprocess of $\mathbf{M}$, that is,

$$G_\alpha^{\mu^-} f(x) := \int_X G_\alpha^{\mu^-}(x,y) f(y)\, m(dy) := E_x\left[\int_0^\zeta \exp\left(-\alpha t - A_t^{\mu^-}\right) f(X_t)\, dt\right]$$

for $f \in \mathcal{B}^+(X)$. Note that

$$h(x) = \int_X G_{-\lambda_1}^{\mu^-}(x,y)\, h(y)\, \mu^+(dy) = G_{-\lambda_1}^{\mu^-}(h\mu^+)(x). \tag{1.16}$$

When $\mu^- = 0$, we simply denote $G_{-\lambda_1}^{\mu^-}(h\mu^+)$ by $G_{-\lambda_1}(h\mu)$.

1.2.3 Ground states of time changed processes

In this subsection, we assume that $\mathbf{M}$ is transient and satisfies Assumption 1.3 (i) and (ii). Let μ be a signed measure on X which can be decomposed into $\mu = \mu^+ - \mu^-$ for some $\mu^+, \mu^- \in \mathcal{K}_\infty$. Then, by the Dirichlet principle (1.9), $\check{\lambda}(\mu^+, \mu^-)$ is the bottom of the spectrum for the $\exp\left(-A_t^{\mu^-}\right)$-subprocess of $\mathbf{M}$ time changed with respect to $A_t^{\mu^+}$. We now make the following assumption:

Assumption 1.6. (Compact embedding of the extended Dirichlet space) *The embedding of* $(\mathcal{F}_e, \mathcal{E})$ *into* $L^2(X; \mu^+)$ *is compact.*

Under this assumption, $\check{\lambda}(\mu^+, \mu^-)$ defined in (1.11) is the principal eigenvalue. Denote by $\check{h}$ the corresponding ground state in $\mathcal{F}_e$. We then see in a similar way to Lemma 2.2 of [57] that

$$\lambda_1(\mu) \geq 0 \iff \check{\lambda}(\mu^+, \mu^-) \geq 1. \tag{1.17}$$

If μ is a signed measure on X which can be decomposed into $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$, then we see from Section 4 of [57] that the ground state $\check{h}$ is a bounded, strictly positive and continuous function on X and that

$$\check{h}(x) = \check{\lambda} \int_X G^{\mu^-}(x,y) \check{h}(y)\, \mu^+(dy),$$

where $G^{\mu^-}(x,y) := G_0^{\mu^-}(x,y)$.

1.3 Branching symmetric Hunt processes

Following [34] and [35], we introduce the notion of branching symmetric Hunt processes. Let $\{p_n(x)\}_{n\geq 0}$, $x \in X$, be a sequence such that

$$0 \leq p_n(x) \leq 1 \quad \text{and} \quad \sum_{n=0}^\infty p_n(x) = 1.$$

For $\mu \in \mathcal{S}$, we denote by Z the random variable of the exponential distribution with rate A_t^{μ}:

$$P_x(t < Z \mid \mathcal{F}_\infty) = \exp\left(-A_t^{\mu}\right).$$

A particle of the branching symmetric Hunt process starts at $x \in X$ according to the law P_x. When $\zeta < Z$, it dies at time ζ. On the other hand, when $Z < \zeta$, it splits into n particles with probability $p_n(X_{Z-})$ at time Z. Then each of these particles starts at X_{Z-} independently according to the law $P_{X_{Z-}}$. Let $X^{(0)} = \{\Delta\}$ and $X^{(1)} = X$. Define the equivalent relation $\sim$ on $X^n = \underbrace{X \times \cdots \times X}_{n}$ as follows; let $\mathbf{x}^n = (x^1, x^2, x^3, \ldots, x^n)$, $\mathbf{y}^n = (y^1, y^2, y^3, \ldots, y^n) \in X^n$. If there exists a permutation σ on $\{1, 2, 3, \cdots, n\}$ such that $y^i = x^{\sigma(i)}$ for all i, then it is denoted by $\mathbf{x}^n \sim \mathbf{y}^n$. Let $X^{(n)} = X^n / \sim$ and $\mathbf{X} = \bigcup_{n=0}^{\infty} X^{(n)}$. When the branching process consists of n particles at time t, they determine a point in $X^{(n)}$. Hence it defines a branching symmetric Hunt process $\overline{\mathbf{M}} = (\mathbf{X}_t, \mathbf{P}_{\mathbf{x}}, \mathcal{G}_t)$ on $\mathbf{X}$ with *motion component* $\mathbf{M}$, *branching rate* μ and *branching mechanism* $\{p_n(x)\}_{n \geq 0}$. Let T be the first splitting time of $\overline{\mathbf{M}}$:

$$\begin{aligned} \mathbf{P}_x(t < T \mid \sigma(X)) &= P_x(t < Z \mid \mathcal{F}_\infty) \\ &= \exp\left(-A_t^{\mu}\right). \end{aligned} \tag{1.18}$$

Denote by Z_t the number of particles of $\overline{\mathbf{M}}$ at time t, that is,

$$Z_t = n \quad \text{if} \quad \mathbf{X}_t = (\mathbf{X}_t^1, \mathbf{X}_t^2, \mathbf{X}_t^3, \ldots \mathbf{X}_t^n) \in X^{(n)}.$$

Define

$$e_0 = \inf\{t > 0 : Z_t = 0\}.$$

Then e_0 is called the *extinction time* of $\overline{\mathbf{M}}$. Set $u_e(x) = \mathbf{P}_x(e_0 < \infty) = \mathbf{P}_x\left(\lim_{t\to\infty} Z_t = 0\right)$. We then say that $\overline{\mathbf{M}}$ *extincts* if $u_e \equiv 1$ on X. Denote by $Z_t(A)$ the number of particles in a set $A \subset X$ at time t and

$$L_A = \sup\{t > 0 : Z_t(A) > 0\}.$$

Set $u_A(x) = \mathbf{P}_x(L_A < \infty) = \mathbf{P}_x\left(\lim_{t\to\infty} Z_t(A) = 0\right)$. We then say that $\overline{\mathbf{M}}$ *extincts locally* if $u_A = 1$ on X for every relatively compact open set A in X.

Let

$$Q(x) = \sum_{n=0}^{\infty} n p_n(x) \quad \text{and} \quad R(x) = \sum_{n=1}^{\infty} n(n-1) p_n(x).$$

We then obtain the following:

Lemma 1.7. *If* $\sup_{x\in X} Q(x) < \infty$, *then*

$$\mathbf{E}_x\left[\sum_{i=1}^{Z_t} f(\mathbf{X}_t^i)\right] = E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) f(X_t); t < \zeta\right] \tag{1.19}$$

for any $f \in \mathcal{B}_b(X)$. *If* $\sup_{x\in X} R(x) < \infty$, *then*

$$\begin{aligned} &\mathbf{E}_x\left[\left(\sum_{i=1}^{Z_t} f(\mathbf{X}_t^i)\right)\left(\sum_{i=1}^{Z_t} g(\mathbf{X}_t^i)\right)\right] = E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) f(X_t) g(X_t); t < \zeta\right] \\ &+ E_x\left[\int_0^{t\wedge\zeta} \exp\left(A_s^{(Q-1)\mu}\right) \mathbf{E}_{X_s}\left[\sum_{i=1}^{Z_{t-s}} f(\mathbf{X}_{t-s}^i)\right] \mathbf{E}_{X_s}\left[\sum_{i=1}^{Z_{t-s}} g(\mathbf{X}_{t-s}^i)\right] dA_s^{R\mu}\right] \end{aligned} \tag{1.20}$$

for any $f, g \in \mathcal{B}_b(X)$.

Proof. Let us denote by $Z_t(m)$ the total number of particles at time t such that each of their trajectories over time interval $[0,t]$ has m branching points, and by

$$\mathbf{X}_t(m) = \left(\mathbf{X}_t^1(m), \mathbf{X}_t^2(m), \cdots, \mathbf{X}_t^{Z_t(m)}(m)\right)$$

the positions of all such particles at time t. Define

$$Z_t(f) = \sum_{i=1}^{Z_t} f(\mathbf{X}_t^i) \quad \text{and} \quad Z_t(m;f) = \sum_{i=1}^{Z_t(m)} f(\mathbf{X}_t^i(m)),$$

respectively, for $f \in \mathcal{B}_b(X)$. Then

$$Z_t(f) = \sum_{m=0}^{\infty} Z_t(m;f).$$

We first show (1.19). It follows from (1.18) that

$$\mathbf{E}_x\left[Z_t(0;f)\right] = E_x\left[\exp\left(-A_t^\mu\right) f(X_t); t < \zeta\right].$$

Since each particle moves independently, the strong Markov property yields that

$$\begin{aligned}
\mathbf{E}_x\left[Z_t(m;f)\right] &= \mathbf{E}_x\left[\mathbf{E}_{\mathbf{X}_T}\left[Z_{t-T}(m-1;f)\right]; T \le t\right] \\
&= \mathbf{E}_x\left[\sum_{i=1}^{Z_T} \mathbf{E}_{\mathbf{X}_T^i}\left[Z_{t-T}(m-1;f)\right]; T \le t\right] \\
&= E_x\left[\int_0^{t\wedge\zeta} \exp\left(-A_s^\mu\right) \mathbf{E}_{X_s}\left[Z_{t-s}(m-1;f)\right] dA_s^{Q\mu}\right].
\end{aligned}$$

By using this relation again, the right hand side above is equal to

$$E_x\left[\int_0^{t\wedge\zeta} \exp\left(-A_s^\mu\right) E_{X_s}\left[\int_0^{(t-s)\wedge\zeta} \exp\left(-A_u^\mu\right) \mathbf{E}_{X_u}\left[Z_{t-s-u}(m-2;f)\right] dA_u^{Q\mu}\right] dA_s^{Q\mu}\right]. \quad (1.21)$$

Since $A_{s+u}^\mu = A_s^\mu + A_u^\mu \circ \theta_s$ and $\zeta = s + \zeta \circ \theta_s$ on $\{s < \zeta\}$, it follows that

$$\begin{aligned}
&E_{X_s}\left[\int_0^{(t-s)\wedge\zeta} \exp\left(-A_u^\mu\right) \mathbf{E}_{X_u}\left[Z_{t-s-u}(m-2;f)\right] dA_u^{Q\mu}\right] \\
&= E_x\left[\int_0^{(t-s)\wedge(\zeta\circ\theta_s)} \exp\left(-A_u^\mu \circ \theta_s\right) \mathbf{E}_{X_u\circ\theta_s}\left[Z_{t-s-u}(m-2;f)\right] dA_u^{Q\mu}\circ\theta_s \,\middle|\, \mathcal{F}_s\right] \\
&= E_x\left[\int_0^{(t-s)\wedge(\zeta-s)} \exp\left(-A_{s+u}^\mu + A_s^\mu\right) \mathbf{E}_{X_{u+s}}\left[Z_{t-s-u}(m-2;f)\right] dA_{s+u}^{Q\mu} \,\middle|\, \mathcal{F}_s\right] \\
&= \exp\left(A_s^\mu\right) E_x\left[\int_s^{t\wedge\zeta} \exp\left(-A_u^\mu\right) \mathbf{E}_{X_u}\left[Z_{t-u}(m-2;f)\right] dA_u^{Q\mu} \,\middle|\, \mathcal{F}_s\right] \quad P_x\text{-a.s. on } \{s < \zeta\}.
\end{aligned}$$

By Fubini's theorem, the term (1.21) is equal to

$$\begin{aligned}
&E_x\left[\int_0^{t\wedge\zeta}\left(\int_s^{t\wedge\zeta}\exp\left(-A_u^\mu\right)\mathbf{E}_{X_u}\left[Z_{t-u}(m-2;f)\right]dA_u^{Q\mu}\right)dA_s^{Q\mu}\right]\\
&=E_x\left[\int_0^{t\wedge\zeta}\exp\left(-A_u^\mu\right)\mathbf{E}_{X_u}\left[Z_{t-u}(m-2;f)\right]\left(\int_0^{u\wedge\zeta}dA_s^{Q\mu}\right)dA_u^{Q\mu}\right]\\
&=E_x\left[\int_0^{t\wedge\zeta}\exp\left(-A_u^\mu\right)\mathbf{E}_{X_u}\left[Z_{t-u}(m-2;f)\right]A_u^{Q\mu}\,dA_u^{Q\mu}\right].
\end{aligned}$$

Hence, by repeating this procedure, we have

$$\mathbf{E}_x\left[Z_t(m;f)\right]=E_x\left[\exp\left(-A_t^\mu\right)\frac{\left(A_t^{Q\mu}\right)^m}{m!}f(X_t);t<\zeta\right],$$

which implies (1.19).

We next show (1.20). Note that

$$\mathbf{E}_x\left[Z_t(0;f)Z_t(0;g)\right]=E_x\left[\exp\left(-A_t^\mu\right)f(X_t)g(X_t);t<\zeta\right]$$

and

$$\mathbf{E}_x\left[Z_t(0;f)Z_t(m;g)\right]=0$$

for $m\geq 1$ because $Z_t(0;f)Z_t(m;g)=0$ by definition. Denote by $Z_t^j(m)$ the total number of children of x^j at time t such that each of their trajectories over time interval $[0,t]$ has m branching points under the law $\mathbf{P}_{\mathbf{x}^\mathbf{n}}$, $\mathbf{x}^n=(x^1,x^2,x^3,\cdots,x^n)\in X^{(n)}$, and by

$$\mathbf{X}_t^j(m)=\left(\mathbf{X}_t^{j,1}(m),\mathbf{X}_t^{j,2}(m),\mathbf{X}_t^{j,3}(m),\cdots\mathbf{X}_t^{j,Z_t^j(m)}(m)\right)$$

the positions of all such particles at time t. Let us define

$$Z_t^j(m;f)=\sum_{i=1}^{Z_t^j(m)}f(\mathbf{X}_t^{j,i}(m)).$$

Then the strong Markov property shows that

$$\begin{aligned}
\mathbf{E}_x\left[Z_t(m;f)Z_t(n;g)\right]&=\mathbf{E}_x\left[\mathbf{E}_{\mathbf{X}_T}\left[Z_t(m-1;f)Z_t(n-1;g)\right];T\leq t\right]\\
&=\mathbf{E}_x\left[\mathbf{E}_{\mathbf{X}_T}\left[\sum_{j=1}^{Z_T}Z_{t-T}^j(m-1;f)Z_{t-T}^j(n-1;g)\right];T\leq t\right]\\
&+\mathbf{E}_x\left[\mathbf{E}_{\mathbf{X}_T}\left[\sum_{1\leq j,k\leq Z_T,j\neq k}Z_{t-T}^j(m-1;f)Z_{t-T}^k(n-1;g)\right];T\leq t\right]
\end{aligned}$$

for $m,n\geq 1$. Moreover, since each particle moves independently, (1.18) yields that the last term above is equal to

$$\mathbf{E}_x\left[\sum_{i=1}^{Z_T}\mathbf{E}_{\mathbf{X}_T^i}\left[Z_{t-T}(m-1;f)Z_{t-T}(n-1;g)\right]\right]$$

$$+\,\mathbf{E}_x\left[\sum_{1\le j,k\le Z_T, j\ne k}\mathbf{E}_{\mathbf{X}_T^j}\left[Z_{t-T}(m-1;f)\right]\mathbf{E}_{\mathbf{X}_T^k}\left[Z_{t-T}(n-1;f)\right]\right]$$

$$= E_x\left[\int_0^{t\wedge\zeta}\exp\left(-A_s^{\mu}\right)\mathbf{E}_{X_s}\left[Z_{t-s}(m-1;f)Z_{t-s}(n-1;g)\right]\,dA_s^{Q\mu}\right]$$

$$+\,E_x\left[\int_0^{t\wedge\zeta}\exp\left(-A_s^{\mu}\right)\mathbf{E}_{X_s}\left[Z_{t-s}(m-1;f)\right]\mathbf{E}_{X_s}\left[Z_{t-s}(n-1;g)\right]\,dA_s^{R\mu}\right].$$

Then, by iterations and Fubini's theorem,

$$\mathbf{E}_x\left[Z_t(m;f)Z_t(m;g)\right] = E_x\left[\exp\left(-A_t^{\mu}\right)\frac{\left(A_t^{Q\mu}\right)^m}{m!}f(X_t)g(X_t);t<\zeta\right]$$

$$+\,E_x\left[\int_0^{t\wedge\zeta}\exp\left(-A_s^{\mu}\right)\sum_{k=1}^{m}\mathbf{E}_{X_s}\left[Z_{t-s}(m-k;f)\right]\mathbf{E}_{X_s}\left[Z_{t-s}(m-k;g)\right]\frac{\left(A_s^{Q\mu}\right)^{k-1}}{(k-1)!}\,dA_s^{R\mu}\right]$$

and

$$\mathbf{E}_x\left[Z_t(m;f)Z_t(n;g)\right]$$

$$= E_x\left[\int_0^{t\wedge\zeta}\exp\left(-A_s^{\mu}\right)\sum_{k=1}^{n}\mathbf{E}_{X_s}\left[Z_{t-s}(m-k;f)\right]\mathbf{E}_{X_s}\left[Z_{t-s}(n-k;g)\right]\frac{\left(A_s^{Q\mu}\right)^{k-1}}{(k-1)!}\,dA_s^{R\mu}\right]$$

for $m>n\ge 1$. Noting that

$$Z_t(f)Z_t(g)=\sum_{m=0}^{\infty}Z_t(m;f)Z_t(m;g)+\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\left(Z_t(m;f)Z_t(n;g)+Z_t(m;g)Z_t(n;f)\right)$$

and using Fubini's theorem, we obtain (1.20). $\square$

1.4 Symmetric α-stable processes

Let $\mathbf{M}^{\alpha}=(\Omega,\mathcal{F},\mathcal{F}_t,\theta_t,X_t,P_x)$, $0<\alpha\le 2$, be a symmetric α-stable process on $\mathbb{R}^d$. Denote by $(\mathcal{E}^{\alpha},\mathcal{F}^{\alpha})$ the Dirichlet form on $L^2(\mathbb{R}^d)$ generated by $\mathbf{M}^{\alpha}$. If $\alpha=2$, then $\mathbf{M}^2$ is the Brownian motion on $\mathbb{R}^d$ and $(\mathcal{E}^2,\mathcal{F}^2)=(\mathbf{D}/2,H^1(\mathbb{R}^d))$, where $H^1(\mathbb{R}^d)$ is the Sobolev space of order one and $\mathbf{D}$ is the Dirichlet integral,

$$\mathbf{D}(u,u)=\int_{\mathbb{R}^d}|\nabla u|^2\,dx,\ \ u\in H^1(\mathbb{R}^d).$$

On the other hand, if $0 < \alpha < 2$, then $\mathbf{M}^\alpha$ is a pure jump process and

$$\mathcal{E}^\alpha(u,u) = \mathcal{A}(d,\alpha) \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dxdy$$

$$\mathcal{F}^\alpha = \left\{ u \in L^2(\mathbb{R}^d) : \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dxdy < \infty \right\},$$

where

$$\mathcal{A}(d,\alpha) = \frac{\alpha 2^{\alpha-3}\Gamma\left(\dfrac{d+\alpha}{2}\right)}{\alpha\pi^{d/2}\Gamma\left(1-\dfrac{\alpha}{2}\right)} \quad \text{and} \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\, dt.$$

If $d > \alpha$, then $\mathbf{M}^\alpha$ is transient and the Green function $G(x,y)$ is given by

$$G(x,y) = \frac{\alpha 2^{1-\alpha}\Gamma\left(\dfrac{d-\alpha}{2}\right)}{\alpha\pi^{d/2}\Gamma\left(\dfrac{\alpha}{2}\right)} \frac{1}{|x-y|^{d-\alpha}}. \tag{1.22}$$

Let $\mathbf{M}^D = (X_t^D, P_x^D)$ be the absorbing symmetric α-stable process on an open set $D \subset \mathbb{R}^d$: set

$$X_t^D = \begin{cases} X_t, & 0 \le t < \tau_D \\ \Delta, & t \ge \tau_D, \end{cases}$$

where τ_D is the exit time of $\mathbf{M}^\alpha$ from D, $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Then the Dirichlet form $(\mathcal{E}^D, \mathcal{F}^D)$ of $\mathbf{M}^D$ is the following:

$$\mathcal{F}^D = \{u \in \mathcal{F}^\alpha : u = 0 \text{ q.e. on } D^c\}$$

$$\mathcal{E}^D(u,u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_D |\nabla u|^2\, dx, & \alpha = 2 \\ \dfrac{1}{2}\mathcal{A}(d,\alpha)\displaystyle\iint_{D\times D\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dxdy & \\ \qquad +\mathcal{A}(d,\alpha)\displaystyle\int_D u(x)^2\left(\int_{D^c}\frac{1}{|x-y|^{d+\alpha}}\, dy\right) dx, & 0 < \alpha < 2 \end{cases}$$

([29, Theorem 4.4.2, Example 4.4.1]). Let $\{p_t^D, t \ge 0\}$ be the Markovian transition semigroup of $\mathbf{M}^D$ given by

$$p_t^D f(x) = E_x^D\left[f\left(X_t^D\right)\right], \quad f \in \mathcal{B}^+(D).$$

By definition,

$$p_t^D f(x) = E_x\left[f(X_t) : t < \tau_D\right].$$

We denote by $p_t^D(x,y)$ the integral kernel of p_t^D,

$$p_t^D f(x) = \int_D p_t^D(x,y)f(y)\, dy.$$

Let $G_\beta^D(x,y)$, $\beta > 0$, be the β-resolvent density of $\mathbf{M}^D$,

$$G_\beta^D(x,y) = \int_0^\infty e^{-\beta t}p_t^D(x,y)\, dt.$$

If $\mathbf{M}^D$ is transient, then the Green function

$$G_0^D(x,y) := \int_0^\infty p_t^D(x,y)\,dt$$

exists for $x \neq y$, and we put $G^D(x,y) = G_0^D(x,y)$. We denote by $\mathcal{K}_\infty^D$ and $\mathcal{S}_\infty^D$ respectively, the classes $\mathcal{K}_\infty$ and $\mathcal{S}_\infty$ associated with $\mathbf{M}^D$ when we need to specify the state space D.

Remark 1.8. (i) We remark that Assumptions 1.3 and 1.5 are satisfied by Brownian motions, symmetric α-stable processes, and that Assumption 1.5 holds for every signed measure $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Clearly Assumption 1.3 is satisfied by Brownian motions and symmetric α-stable processes. That Assumption 1.5 holds for symmetric α-stable processes is proved [55]. We also note that Assumption 1.6 is satisfied by transient Brownian motions and symmetric α-stable processes. That Assumption 1.6 holds for symmetric α-stable processes is proved [57].

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(\mathbb{R}^d)$ and $\mathbf{M}$ the associated symmetric Hunt process. If $(\mathcal{E}_1,\mathcal{F})$ is comparable to that of the symmetric α-stable process, then, by applying the same argument as in [55], we can show that the embedding of $(\mathcal{F},\mathcal{E}_1)$ to $L^2(\mathbb{R}^d;\mu)$ is compact for any $\mu \in \mathcal{K}_\infty$.

For instance, we consider stable-like processes on $\mathbb{R}^d$ in the sense of [17]; let $c(x,y)$ be a symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ which is bounded between two positive constants $c_2 > c_1 > 0$, that is,

$$c_1 \leq c(x,y) \leq c_2, \quad \text{a.e. } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Fix $0 < \alpha < 2$ and define

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} c(x,y)\,dxdy$$

$$\mathcal{F} = \left\{ u \in L^2(\mathbb{R}^d) : \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dxdy < \infty \right\}.$$

Since $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$, there exists an associated symmetric Hunt process on $\mathbb{R}^d$, which is called the *α-stable-like process*. Clearly the Dirichlet form $(\mathcal{E},\mathcal{F})$ is comparable to that of the symmetric α-stable process. Moreover, it is proved in [17, Theorem 4.14] that the α-stable-like process on $\mathbb{R}^d$ admits a Hölder continuous transition density which is comparable to that of the symmetric α-stable process. These facts imply that stable-like processes on $\mathbb{R}^d$ fulfill Assumptions 1.3 and 1.5. In addition, if $d > \alpha$, then they also fulfill Assumption 1.6. Note that the class $\mathcal{K}_\infty$ of the α-stable-like process on $\mathbb{R}^d$ is identified with that of the symmetric α-stable process on $\mathbb{R}^d$.

We note that relativistic α-stable processes also satisfy Assumptions 1.3 and 1.5. Let us denote by $(\mathcal{E}^\alpha,\mathcal{F}^\alpha)$ and $(\mathcal{R}^\alpha,\mathcal{D}(\mathcal{R}^\alpha))$ the Dirichlet forms on $L^2(\mathbb{R}^d;dx)$ respectively generated by the symmetric α-stable process and the relativistic α-stable process. Since $(\mathcal{R}_1^\alpha,\mathcal{D}(\mathcal{R}^\alpha))$ is comparable to $(\mathcal{E}_1^\alpha,\mathcal{F}^\alpha)$ by (3.7) of [20], Assumption 1.5 holds for relativistic α-stable processes by applying the arguments in [55, Section 2] to $(\mathcal{R}_1^\alpha,\mathcal{D}(\mathcal{R}^\alpha))$.

(ii) Let M be a simply connected, complete and non-compact Riemannian manifold and consider the Brownian motion on M. Denote by $(\mathcal{E},\mathcal{F})$ the associated regular Dirichlet form on $L^2(M;V)$:

$$\mathcal{E}(u,u) = \frac{1}{2}\int_M |\nabla u|^2\,dV$$

$$\mathcal{F} = \text{the closure of } C_0^\infty(M) \text{ with respect to } \mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{L^2(M;V)},$$

where V is the Riemannian volume of M. We then see in a similar way to [56, Section 3] that Assumption 1.5 is satisfied. Moreover, if the Brownian motion on M is transient, then Assumption 1.6 is also satisfied. On the other hand, we can find in [22, Section 5] some sufficient conditions for the Brownian motion on M to satisfy Assumption 1.3.

Remark 1.9. Recall that $\mathbf{M}^D$ is the absorbing symmetric α-stable process on an open set D in $\mathbb{R}^d$. Assume that $\mathbf{M}^D$ is transient. We now show that, if the support of a measure $\nu \in \mathcal{S}_\infty^D$ is compact, then ν belongs to $\mathcal{S}_\infty\left(G_\beta^D\right)$ for any $\beta > 0$. Fix a measure $\nu \in \mathcal{S}_\infty^D$ with compact support and put $F = \mathrm{supp}[\nu]$. Let O be a bounded $C^{1,1}$ *domain* in D such that $F \subset O$. Here we say that O is a $C^{1,1}$ domain, if for any $x \in \partial O$, there exists a positive constant $r > 0$ such that $B_x(r) \cap \partial O$ is the graph of a function whose first derivatives are Lipschitz continuous, where $B_x(r) = \left\{y \in \mathbb{R}^d : |x-y| \le r\right\}$. Since $G^D(x,y) \le G(x,y)$, Corollary 1.3 of [18] implies that

$$G^O(x,y) \le G^D(x,y) \le CG^O(x,y)$$

for any $x, y \in F$, where $C \ge 1$ is some positive constant depending on F. Furthermore, since

$$\sup_{(x,z)\in O\times O\setminus\triangle} \int_O \frac{G^O(x,y)G^O(y,z)}{G^O(x,z)}\, dy < \infty$$

by Theorem 1.8 of [18], it follows from Theorem 5.3 of [14] and Lemma 3.3 of [52] that

$$G_\beta^O(x,y) \le G^O(x,y) \le CG_\beta^O(x,y)$$

for any $x, y \in O$, which leads us to that

$$G_\beta^D(x,y) \le G^D(x,y) \le CG_\beta^D(x,y)$$

for any $x, y \in F$. Here the constants C above are different and depend on β. Therefore, for any nonnegative Borel function f on D,

$$\begin{aligned} \sup_{(x,z)\in D\times D\setminus\triangle} \int_D \frac{G_\beta^D(x,y)G_\beta^D(y,z)}{G_\beta^D(x,z)} f(y)\,\nu(dy) &= \sup_{(x,z)\in F\times F\setminus\triangle} \int_F \frac{G_\beta^D(x,y)G_\beta^D(y,z)}{G_\beta^D(x,z)} f(y)\,\nu(dy) \\ &\le C \sup_{(x,z)\in F\times F\setminus\triangle} \int_F \frac{G^D(x,y)G^D(y,z)}{G^D(x,z)} f(y)\,\nu(dy). \end{aligned} \tag{1.23}$$

Note that the following $3G$-inequality holds locally for $G^D(x,y)$:

$$\frac{G^D(x,y)G^D(y,z)}{G^D(x,z)} \le C(G^D(x,y) + G^D(y,z)), \quad (x,z) \in F \times F \setminus \triangle,$$

where C is a constant depending on F. Thereby the right hand side of (1.23) is not greater than

$$2C \sup_{x\in F} \int_F G^D(x,y) f(y)\,\nu(dy) = 2C \sup_{x\in D} \int_D G^D(x,y) f(y)\,\nu(dy),$$

which shows that ν belongs to $\mathcal{S}_\infty\left(G_\beta^D\right)$.

Let μ be a signed measure on D which can be decomposed into $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty^D - \mathcal{K}_\infty^D$ such that the supports of μ^+ and μ^- are compact. Define

$$\lambda_1(\mu; D) = \inf \left\{ \mathcal{E}^D(u,u) - \int_D u^2\, d\mu \ : u \in C_0^\infty(D),\ \int_D u^2\, dx = 1 \right\}.$$

Assume that $\lambda_1 := \lambda_1(\mu; D) < 0$ and let h be the corresponding ground state with normalization $\int_D h^2\, dx = 1$. Since μ^+ and μ^- belong to $\mathcal{S}_\infty\left(G_{-\lambda}^D\right)$ as discussed above, we can show that, by the same way as in Section 4 of [57],

$$C^{-1} G_{-\lambda_1}^D(o,x) \le h(x) \le C G_{-\lambda_1}^D(o,x), \quad x \in D \setminus K \tag{1.24}$$

for a compact set $K \subset D$ and a fixed point $o \in K$, where $C \ge 1$ is some positive constant depending on K. On the other hand, if $D = \mathbb{R}^d$, then

$$h(x) \le C \exp\left(-\sqrt{-2\lambda_1}|x|\right), \qquad |x| \ge 1 \tag{1.25}$$

for $\alpha = 2$ and

$$\frac{C^{-1}}{|x|^{d+\alpha}} \le h(x) \le \frac{C}{|x|^{d+\alpha}}, \qquad |x| \ge 1 \tag{1.26}$$

for $0 < \alpha < 2$ by (II.18) of [12].

Chapter 2

Extinction of branching symmetric Markov processes

In this chapter, we give a criterion for extinction or local extinction of branching symmetric Hunt processes in terms of the principal eigenvalues for time changed processes of symmetric Hunt processes. Here the branching rates and the branching mechanisms can be state-dependent. In particular, the branching rates can be singular with respect to the Lebesgue measure. We apply this criterion to branching Brownian motions and branching symmetric α-stable processes.

2.1 Extinction and local extinction

Let X be a locally compact separable metric space and m a positive smooth Radon measure on X with full support. Let $\mathbf{M} = (X_t, P_x)$ be an m-symmetric Hunt process on X. Throughout this section, we assume that $\mathbf{M}$ is transient and satisfies Assumption 1.3 (i) and (ii), that is, $\mathbf{M}$ is irreducible and the semigroup satisfies the strong Feller property.

Let $\overline{\mathbf{M}} = (\mathbf{X}_t, \mathbf{P}_{\mathbf{x}})$ be a branching symmetric Hunt process with motion component $\mathbf{M}$, branching rate $\mu \in \mathcal{K}_\infty$ and branching mechanism $\{p_n(x)\}_{n\geq 0}$. We first consider the extinction problem of $\overline{\mathbf{M}}$. Let

$$F(u)(\cdot) = \sum_{n=0}^{\infty} p_n(\cdot)u(\cdot)^n.$$

We characterize the function $u_e(x) := \mathbf{P}_x(e_0 < \infty)$ as a solution to the equation as follows.

Proposition 2.1. *The function u_e is a minimal solution to*

$$u(x) = E_x\left[\exp\left(-A^\mu_\zeta\right); \zeta < \infty\right] + E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right) F(u)(X_t)\, dA^\mu_t\right], \quad 0 \leq u \leq 1. \tag{2.1}$$

Proof. The strong Markov property of $\overline{\mathbf{M}}$ implies that

$$\begin{aligned} u_e(x) &= \mathbf{P}_x(e_0 = \zeta < T,\ e_0 < \infty) + \mathbf{P}_x(e_0 = T < \zeta,\ e_0 < \infty) \\ &\quad + \mathbf{P}_x(T < e_0 \wedge \zeta,\ e_0 < \infty) \\ &= \mathbf{P}_x(\zeta < T,\ \zeta < \infty) + \mathbf{P}_x(e_0 = T < \zeta,\ e_0 < \infty) \\ &\quad + \mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_T}(e_0 < \infty); T < e_0 \wedge \zeta\right]. \end{aligned}$$

Since

$$\mathbf{P}_x(\zeta < T,\ \zeta < \infty) = E_x\left[\exp\left(-A^\mu_\zeta\right); \zeta < \infty\right],$$

$$\mathbf{P}_x(e_0 = T < \zeta,\ e_0 < \infty) = E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right) p_0(X_t)\, dA^\mu_t\right],$$

$$\mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_T}(e_0 < \infty); T < e_0 \wedge \zeta\right] = E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right) \sum_{n=1}^\infty p_n(X_t) u_e(X_t)^n\, dA^\mu_t\right],$$

the function u_e is a solution to (2.1).

Let $R = \inf\{t > 0 : Z_t \neq Z_0\}$ and $S = R \wedge T$. Define

$$\begin{aligned} &S_0 = 0 \\ &S_k = S_{k-1} + S \circ \theta_{S_{k-1}},\ k \geq 1, \end{aligned}$$

where θ_t is the shift of paths for $\overline{\mathbf{M}}$. If $Z_{S_k} = 0$ for some $k \geq 1$, then we define $S_l = S_k$ for all $l \geq k$. Let $u_k(x) = \mathbf{P}_x(Z_{S_k} = 0,\ e_0 < \infty)$. Then $u_0 \equiv 0$ and

$$\begin{aligned} u_k(x) = \mathbf{P}_x(\zeta < T,\ \zeta < \infty) &+ \mathbf{P}_x\left(e_0 = T < \zeta,\ e_0 < \infty\right) \\ &+ \mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_T}\left(Z_{S_{k-1}} = 0,\ e_0 < \infty\right); T < e_0 \wedge \zeta\right]. \end{aligned} \tag{2.2}$$

Let $\mathbf{x}^n = (x^1, x^2, x^3, \cdots, x^n) \in X^{(n)}$. Since

$$\mathbf{P}_{\mathbf{x}^n}\left[Z_{S_{k-1}} = 0,\ e_0 < \infty\right] \leq \prod_{i=1}^n \mathbf{P}_{x^i}\left[Z_{S_{k-1}} = 0,\ e_0 < \infty\right],$$

the last term of the right hand side of (2.2) is not greater than

$$\begin{aligned} &\mathbf{E}_x\left[\prod_{i=1}^{Z_T} \mathbf{P}_{\mathbf{X}^i_T}\left(Z_{S_{k-1}} = 0,\ e_0 < \infty\right); T < e_0 \wedge \zeta\right] \\ &= E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right) \sum_{n=1}^\infty p_n(X_t) u_{k-1}(X_t)^n\, dA^\mu_t\right], \end{aligned}$$

and thus

$$u_k(x) \leq E_x\left[\exp\left(-A^\mu_\zeta\right) : \zeta < \infty\right] + E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right) F(u_{k-1})(X_t)\, dA^\mu_t\right]. \tag{2.3}$$

Suppose that a function v is also a solution to (2.1). On account of (2.3), $u_k \leq v$ for any $k \geq 1$ by induction, which implies that $\lim_{k\to\infty} u_k = u_e \leq v$. □

Lemma 2.2. *Any solution to* (2.1) *is finely continuous.*

Proof. Let u be a solution to (2.1). Then the Markov property of $\mathbf{M}$ yields that

$$\begin{aligned} u(X_t) &= E_{X_t}\left[\exp\left(-A^\mu_\zeta\right); \zeta < \infty\right] + E_{X_t}\left[\int_0^\zeta \exp\left(-A^\mu_s\right) F(u)(X_s)\, dA^\mu_s\right] \\ &= E_x\left[\exp\left(-A^\mu_{\zeta\circ\theta_t} \circ \theta_t\right); \zeta \circ \theta_t < \infty \,\middle|\, \mathcal{F}_t\right] \\ &\quad + E_x\left[\int_0^{\zeta\circ\theta_t} \exp\left(-A^\mu_s \circ \theta_t\right) F(u)(X_s \circ \theta_t)\, dA^\mu_s \circ \theta_t \,\middle|\, \mathcal{F}_t\right] \quad P_x\text{-a.s. on } \{t < \zeta\} \end{aligned}$$

for all $x \in X$. Since it holds that $A^\mu_{s+t} = A^\mu_t + A^\mu_s \circ \theta_t$ and that $\zeta = t + \zeta \circ \theta_t$ P_x-a.s. on $\{t < \zeta\}$, the right hand side above is equal to

$$\begin{aligned}
&E_x\left[\exp\left(-A^\mu_{t+\zeta\circ\theta_t} + A^\mu_t\right);\zeta<\infty\,\middle|\,\mathcal{F}_t\right] + E_x\left[\int_0^{\zeta-t}\exp\left(-A^\mu_{s+t}+A^\mu_t\right)F(u)(X_{s+t})\,dA^\mu_{s+t}\,\middle|\,\mathcal{F}_t\right]\\
&= \exp\left(A^\mu_t\right)E_x\left[\exp\left(-A^\mu_\zeta\right);\zeta<\infty\,\middle|\,\mathcal{F}_t\right] + \exp\left(A^\mu_t\right)E_x\left[\int_t^{\zeta}\exp\left(-A^\mu_s\right)F(u)(X_s)\,dA^\mu_s\,\middle|\,\mathcal{F}_t\right]\\
&= \exp\left(A^\mu_t\right)E_x\left[\exp\left(-A^\mu_\zeta\right);\zeta<\infty\,\middle|\,\mathcal{F}_t\right] + \exp\left(A^\mu_t\right)E_x\left[\int_0^{\zeta}\exp\left(-A^\mu_s\right)F(u)(X_s)\,dA^\mu_s\,\middle|\,\mathcal{F}_t\right]\\
&\qquad - \exp\left(A^\mu_t\right)\int_0^t \exp\left(-A^\mu_s\right)F(u)(X_s)\,dA^\mu_s.
\end{aligned}$$

By noting that the right hand side above is right continuous by the right continuity of the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, the function u is finely continuous by [9, §4 Theorem 4.8]. □

Lemma 2.3. *If $P_x(\zeta < \infty) < 1$ for $x \in X$, then $u_e(x) < 1$, that is, the process $\overline{\mathbf{M}}$ does not extinct.*

Proof. Since $u_e \leq 1$ and

$$E_x\left[\exp\left(-A^\mu_\zeta\right);\zeta=\infty\right] > 0,$$

(2.1) implies that

$$\begin{aligned}
u_e(x_0) &\leq E_x\left[\exp\left(-A^\mu_\zeta\right);\zeta<\infty\right] + E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right)F(1)(X_t)\,dA^\mu_t\right]\\
&= 1 - E_x\left[\exp\left(-A^\mu_\zeta\right);\zeta=\infty\right] < 1.
\end{aligned}$$

□

Recall that

$$Q(x) = F'(1)(x) = \sum_{n=1}^\infty np_n(x)$$

and suppose that $\sup_{x\in X} Q(x) < \infty$. Put

$$\check{\lambda}(\mu, Q) = \inf\left\{\mathcal{E}(u,u) + \int_X u^2\,d\mu : f \in \mathcal{F},\ \int_X f^2\,Qd\mu = 1\right\}. \tag{2.4}$$

We then have

Theorem 2.4. *Assume that $P_x(\zeta < \infty) = 1$ for all $x \in X$ and that the branching rate μ belongs to the class $\mathcal{S}_\infty$. Then, under Assumption* 1.6, *$\overline{\mathbf{M}}$ extincts if and only if $\check{\lambda}(\mu, Q) \geq 1$.*

Proof. Let $\check{\lambda} := \check{\lambda}(\mu, Q)$. First suppose that $\check{\lambda} \geq 1$. Let u be a solution to (2.1) and denote by σ_A the hitting time of a set A in X, $\sigma_A = \inf\{t > 0 : X_t \in A\}$. Let $O = \{x \in X : u(x) < 1\}$ and assume that $P_x(\sigma_{F^\mu \cap O} < \infty) > 0$ for all $x \in X$, where F^μ is the fine support of the measure

μ defined in (1.8). Since u is finely continuous by Lemma 2.2 and $u - u^n < (n-1)(1-u)$ on O for $n \geq 2$, it follows from (2.1) and the assumption on the lifetime that

$$\begin{aligned} u(x) &= E_x \left[\exp\left(-A^\mu_\zeta\right) \right] + E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) u(X_t)\, dA^\mu_t \right] \\ &\quad + E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) \sum_{n=0}^\infty p_n(X_t)(u(X_t)^n - u(X_t))\, dA^\mu_t \right] \\ &> E_x \left[\exp\left(-A^\mu_\zeta\right) \right] + E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) u(X_t)\, dA^\mu_t \right] \\ &\quad - E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) \sum_{n=0}^\infty (n-1) p_n(X_t)(1 - u(X_t))\, dA^\mu_t \right] \end{aligned}$$

for all $x \in X$. Let $v = 1 - u$. Then the right hand side above is equal to

$$\begin{aligned} & E_x \left[\exp\left(-A^\mu_\zeta\right) \right] + E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) (1 - v(X_t))\, dA^\mu_t \right] \\ &\quad - E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) \sum_{n=0}^\infty (n-1) p_n(X_t) v(X_t)\, dA^\mu_t \right] \\ &= 1 - E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) v(X_t)\, dA^{Q\mu}_t \right]. \end{aligned}$$

Hence

$$0 \leq v(x) < G^{\mu, Q\mu} v(x), \tag{2.5}$$

where $G^{\mu,Q\mu}$ is the generalized resolvent defined by

$$G^{\mu,Q\mu} f(x) = E_x \left[\int_0^\zeta \exp\left(-A^\mu_t\right) f(X_t)\, dA^{Q\mu}_t \right]$$

for any measurable function f in X such that the right hand side of the expression makes sense.

Let $\check{h}$ be the ground state corresponding to $\check{\lambda}$, that is, the function attaining the infimum of the right hand side of (2.4). Then the function $\check{h}$ is bounded, strictly positive and continuous on X as mentioned in Subsection 1.2.3 before. Since the branching rate μ belongs to the class $\mathcal{S}_\infty$, it follows that, for a compact set $K \subset X$ and a fixed point $o \in K$,

$$\check{h}(x) \leq C G(o, x), \quad x \in X \setminus K, \tag{2.6}$$

where $G(x, y)$ is the Green function of $\mathbf{M}$ and C is a constant depending on K. Thus,

$$\begin{aligned} \int_{X \setminus K} \check{h}(y)\, \mu(dy) &\leq C \int_{X \setminus K} G(o, y)\, \mu(dy) \\ &\leq C \sup_{x \in X} \int_{X \setminus K} G(x, y)\, \mu(dy) < \infty \end{aligned}$$

by the fact that $\mathcal{S}_\infty \subset \mathcal{K}_\infty$ and the definition of $\mathcal{K}_\infty$. Noting that

$$\int_K \check{h}(x)\, \mu(dx) \leq \|\check{h}\|_\infty \mu(K) < \infty,$$

we see that

$$\int_X \check{h}(x)\,\mu(dx) = \int_K \check{h}(x)\,\mu(dx) + \int_{X\setminus K} \check{h}(x)\,\mu(dx) < \infty.$$

Since

$$\check{h}(x) = \check{\lambda} G^{\mu,Q\mu}\check{h}(x), \tag{2.7}$$

the inequality (2.5) shows that

$$\begin{aligned}\int_X \check{h}(x)v(x)\,Q(x)\mu(dx) &= \check{\lambda}\int_X G^{\mu,Q\mu}\check{h}(x)v(x)\,Q(x)\mu(dx)\\ &= \check{\lambda}\int_X \check{h}(x)G^{\mu,Q\mu}v(x)\,Q(x)\mu(dx)\\ &> \check{\lambda}\int_X \check{h}(x)v(x)\,Q(x)\mu(dx),\end{aligned}$$

where the second equality holds because of the $Q\mu$-symmetry of $G^{\mu,Q\mu}$ (see Theorem 2.2 (iv) of [2]). This contradicts the assumption that $\check{\lambda} \geq 1$. Hence $P_x(\sigma_{F^\mu\cap O} < \infty) = 0$ for some $x \in X$, which implies that $P_x(\sigma_{F^\mu\cap O} < \infty) = 0$ for all $x \in X$ because of the irreducibility of $\mathbf{M}$. Accordingly the equation (2.1) yields that $u \equiv 1$ on X, and thus $u_e \equiv 1$ on X by Proposition 2.1.

Next suppose that $\check{\lambda} < 1$. Choose a positive constant β such that $\check{\lambda} < \beta < 1$ and a positive constant ε such that $0 < \varepsilon < 1$ and $F'(1-\varepsilon) \geq \beta F'(1) = \beta Q(x)$. Let δ be a positive constant such that $\delta\sup_{x\in X}\check{h}(x) \leq \varepsilon$ and $w(x) = 1 - \delta\check{h}(x)$. Then

$$\begin{aligned}&E_x\left[\exp\left(-A^\mu_\zeta\right)\right] + E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right)F(w)(X_t)\,dA^\mu_t\right]\\ &= 1 - E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right)(F(1)-F(w))(X_t)\,dA^\mu_t\right]\\ &= 1 - E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right)F'(\gamma)(X_t)(1-w(X_t))\,dA^\mu_t\right],\end{aligned} \tag{2.8}$$

where γ is a function satisfying $1-\varepsilon < w(x) < \gamma(x) < 1$ for all $x \in X$. Since

$$F'(\gamma)(x) \geq F'(1-\varepsilon)(x) \geq \beta Q(x),$$

the right hand side of (2.8) is not greater than

$$\begin{aligned}1 - \beta\delta G^{\mu,Q\mu}\check{h}(x) &= 1 - \frac{\beta}{\check{\lambda}}\delta\check{h}(x)\\ &< 1 - \delta\check{h}(x) = w(x)\end{aligned}$$

by (2.7) and the relation $\beta > \check{\lambda}$. Thus,

$$E_x\left[\exp\left(-A^\mu_\zeta\right)\right] + E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right)F(w)(X_t)\,dA^\mu_t\right] < w(x). \tag{2.9}$$

On account of (2.3) and (2.9), we see that $u_k < w$ for any $k \geq 1$ by induction. Hence $\lim_{k\to\infty} u_k = u_e \leq w < 1$ on X. $\square$

Remark 2.5. Assume that $P_x(\zeta < \infty) = 1$ for all $x \in X$ and that $\mu \in \mathcal{S}_\infty$. Recall that

$$\check{\lambda}(\mu) = \inf \left\{ \mathcal{E}(u,u) : u \in \mathcal{F}, \int_X u^2 \, d\mu = 1 \right\}.$$

If $Q(x) \equiv Q$, then

$$\check{\lambda}(\mu, Q) \geq 1 \iff \check{\lambda}(\mu) \geq Q - 1.$$

This fact says that if $Q \leq 1$, then, under Assumption 1.6, $\overline{\mathbf{M}}$ extincts for any branching rate μ in $\mathcal{S}_\infty$.

Let $\mathbf{M}^D$ be the part of the process $\mathbf{M}$ on an open set D in X and $(\mathcal{E}^D, \mathcal{F}^D)$ the associated Dirichlet form on $L^2(D)$. Denote by $\mathcal{S}_\infty^D$ the class $\mathcal{S}_\infty$ associated with $\mathbf{M}^D$. Then the following is known:

Lemma 2.6. ([54, Lemma 4.5]) *Let D be an open set in X such that* $\mathrm{Cap}(X \setminus D) > 0$ *and $\mu, \nu \in \mathcal{S}_\infty^D$. Then under Assumption* 1.6*, it holds that $\check{\lambda}(\mu, \nu; D) > \check{\lambda}(\mu, \nu)$, where*

$$\check{\lambda}(\mu, \nu; D) = \inf \left\{ \mathcal{E}^D(u,u) + \int_D u^2 \, d\nu : u \in \mathcal{F}^D, \int_D u^2 \, d\mu = 1 \right\}.$$

We denote by $\overline{\mathbf{M}^D}$ the branching symmetric Hunt process such that the motion component is $\mathbf{M}^D$ and the branching rate is a measure μ belonging to $\mathcal{S}_\infty^D$. Combining Theorem 2.4 with Lemma 2.6 yields the following:

Corollary 2.7. *Let D be an open set in X such that $P_x(\tau_D < \infty) = 1$ for all $x \in D$, where $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Then under Assumption* 1.6*, the branching process $\overline{\mathbf{M}^D}$ extincts if $\check{\lambda}(\mu, Q) \geq 1$.*

Consider a branching diffusion process on a metric space. Then it is known that the expectation of the number of branches hitting a closed set coincides with the expectation of the Feynman-Kac functional (see [38]). This relation also holds for branching symmetric Hunt processes on X. Combining this fact with Theorem 1.2, Takeda [54] showed the following:

Theorem 2.8. ([54, Theorem 1.2]) *Let N_K be the number of branches of $\overline{\mathbf{M}}$ ever hitting a closed set K in X. Then, under Assumption* 1.6*, it holds that*

$$\sup_{x \in X \setminus K} \mathbf{E}_x \left[N_K \right] < \infty \iff \lambda(\mu, Q; X \setminus K) > 1$$

for any closed set K in X with $\mathrm{Cap}(K) > 0$*, where $\check{\lambda}(\mu, Q; X \setminus K) := \check{\lambda}(Q\mu, \mu; X \setminus K)$.*

We note that, although Lemma 2.6 and Theorem 2.8 are proved in [54] for branching symmetric α-stable processes, the arguments there also work for more general branching processes considered in this section.

Let D be an open set in X such that $P_x(\tau_D < \infty) = 1$ for all $x \in X$. Assume that the branching rate μ belongs to $\mathcal{S}_\infty^D$. Then, under Assumption 1.6, Theorem 2.4 and Theorem 2.8 show the following:

$$\begin{aligned}
\check{\lambda}(\mu, Q; D) > 1 &\Rightarrow \mathbf{P}_x(e_0 < \infty) \equiv 1, \sup_{x \in D} \mathbf{E}_x \left[N_{X \setminus D} \right] < \infty \\
\check{\lambda}(\mu, Q; D) = 1 &\Rightarrow \mathbf{P}_x(e_0 < \infty) \equiv 1, \sup_{x \in D} \mathbf{E}_x \left[N_{X \setminus D} \right] = \infty \\
\check{\lambda}(\mu, Q; D) < 1 &\Rightarrow \mathbf{P}_x(e_0 < \infty) < 1, \sup_{x \in D} \mathbf{E}_x \left[N_{X \setminus D} \right] = \infty.
\end{aligned}$$

The next lemma says that

$$\{e_0 = \infty\} = \left\{\lim_{t\to\infty} Z_t = \infty\right\} \quad \mathbf{P}_x\text{-}a.s.$$

Lemma 2.9. *If $P_x(\zeta < \infty) = 1$ for all $x \in X$, then*

$$\mathbf{P}_x\left(\lim_{t\to\infty} Z_t = 0 \quad or \quad \lim_{t\to\infty} Z_t = \infty\right) = 1, \quad x \in D.$$

Proof. We first show that

$$\mathbf{P}_{\mathbf{x}^k}(Z_t = k,\ \forall t \geq 0) = 0,\ \mathbf{x}^k \in X^{(k)} \tag{2.10}$$

for any $k \geq 1$. Note that it suffices to consider the case $k = 1$. Define

$$\begin{aligned} T_1 &= T \\ T_n &= T_{n-1} + T \circ \theta_{T_{n-1}} \end{aligned}$$

for $n \geq 1$. Then T_n denotes the nth branching time of $\overline{\mathbf{M}}$. Let B be the total number of particle splits,

$$B = \begin{cases} 0, & \text{if } \ T = \infty \\ \sup\{n \geq 1 : T_n < \infty\}, & \text{otherwise,} \end{cases}$$

and let

$$s_n(x) = \mathbf{P}_x\left(Z_t = 1,\ \forall t \geq 0,\ B = n\right).$$

Since

$$s_0(x) = E_x\left[\exp\left(-A^{\mu}_{\zeta}\right); \zeta = \infty\right] = 0$$

by the assumption on the lifetime, it follows that

$$\begin{aligned} s_n(x) &= \mathbf{E}_x\left(\mathbf{P}_{\mathbf{X}_T}(Z_t = 1,\ \forall t \geq 0,\ B = n-1); T < e_0 \wedge \zeta\right) \\ &= E_x\left[\int_0^{\zeta} \exp\left(-A^{\mu}_t\right) p_1(X_t) s_{n-1}(X_t)\, dA^{\mu}_t\right] = 0 \end{aligned}$$

by induction. Consequently,

$$\mathbf{P}_x\left(Z_t = 1 \text{ for all } t \geq 0,\ B < \infty\right) = \sum_{n=0}^{\infty} s_n(x) = 0$$

for all $x \in X$. Let

$$t(x) = \mathbf{P}_x\left(Z_t = 1 \text{ for all } t \geq 0,\ B = \infty\right).$$

Then

$$\begin{aligned} t(x) &= E_x\left[\int_0^{\zeta} \exp\left(-A^{\mu}_t\right) p_1(X_t) t(X_t)\, dA^{\mu}_t\right] \\ &= E_x\left[\int_0^{\zeta} \exp\left(-A^{\mu}_t\right) E_{X_t}\left[\int_0^{\zeta} \exp\left(-A^{\mu}_s\right) t(X_s)\, dA^{p_1\mu}_s\right] dA^{p_1\mu}_t\right] \\ &= E_x\left[\int_0^{\zeta} \exp\left(-A^{\mu}_t\right) E_x\left[\int_0^{\zeta\circ\theta_t} \exp\left(-A^{\mu}_s \circ \theta_t\right) t(X_s \circ \theta_t)\, dA^{p_1\mu}_s \circ \theta_t \,\middle|\, \mathcal{F}_t\right] dA^{p_1\mu}_t\right] \\ &= E_x\left[\int_0^{\zeta} \exp\left(-A^{\mu}_t\right) \left(\int_t^{\zeta} \exp\left(-A^{\mu}_s + A^{\mu}_t\right) t(X_s)\, dA^{p_1\mu}_s\right) dA^{p_1\mu}_t\right]. \end{aligned}$$

By Fubini's theorem, the last term above is equal to

$$E_x\left[\int_0^{\zeta} \exp\left(-A_s^{\mu}\right) t(X_s) A_s^{p_1\mu} dA_s^{p_1\mu}\right].$$

By repeating this procedure, it follows that

$$t(x) = E_x\left[\int_0^{\zeta} \exp\left(-A_t^{\mu}\right) \frac{(A_t^{p_1\mu})^n}{n!} t(X_t)\, dA_t^{p_1\mu}\right]$$

for any positive integer n. Since

$$\begin{aligned}\sum_{n=0}^{\infty} t(x) &= E_x\left[\int_0^{\zeta} \exp\left(-A_t^{(1-p_1)\mu}\right) t(X_t)\, dA_t^{p_1\mu}\right] \\ &\leq E_x\left[\int_0^{\zeta} dA_t^{\mu}\right] \leq \sup_{x\in D} E_x\left[A_\zeta^{\mu}\right] < \infty\end{aligned}$$

by (1.10), it follows that $t \equiv 0$ on X, which implies (2.10).

We next show that the probability that Z_t equals k infinitely often is 0 for each positive integer k. For a positive integer k, let

$$\begin{aligned}U = U_1 &= \inf\left\{t > 0 : Z_t = k\right\} \\ V_1 &= U_1 + R \circ \theta_{U_1} \\ U_n &= V_{n-1} + U \circ \theta_{V_{n-1}} \\ V_n &= U_n + R \circ \theta_{U_n},\end{aligned}$$

and $\inf\{\emptyset\} = \infty$, where $R = \inf\{t > 0 : Z_t \neq Z_0\}$. Then

$$\begin{aligned}\mathbf{P}_x\left(Z_t = k \text{ infinitely often}\right) &= \mathbf{P}_x\left(\bigcap_{n\geq 1}\{U_n < \infty\}\right) \\ &= \lim_{n\to\infty} \mathbf{P}_x\left(U_n < \infty\right).\end{aligned}$$

The strong Markov property of $\overline{\mathbf{M}}$ implies that

$$\begin{aligned}\mathbf{P}_x\left(U_2 < \infty\right) &= \mathbf{P}_x\left(V_1 + U \circ \theta_{V_1} < \infty\right) \\ &= \mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_{V_1}}\left(U < \infty\right); V_1 < \infty\right].\end{aligned}$$

By the definition of V_1, the right hand side above is equal to

$$\begin{aligned}&\mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_{U_1 + R\circ\theta_{U_1}}}\left(U < \infty\right); U_1 + R \circ \theta_{U_1} < \infty\right] \\ &= \mathbf{E}_x\left[\mathbf{E}_{\mathbf{X}_{U_1}}\left[\mathbf{P}_{\mathbf{X}_R}(U < \infty); R < \infty\right]; U_1 < \infty\right] \\ &= \mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_{U_1}}\left(R + U \circ \theta_R < \infty\right); U_1 < \infty\right].\end{aligned}$$

Let $\gamma = \exp\left(-\sup_{x\in X} E_x\left[A_\zeta^{\mu}\right]\right)$. Then $\gamma > 0$ by (1.10) and

$$\mathbf{P}_x\left(\zeta < T\right) = E_x\left[\exp\left(-A_\zeta^{\mu}\right)\right] \geq \gamma$$

by the assumption on the lifetime and Jensen's inequality. As a direct calculation yields that

$$\begin{aligned}\mathbf{P}_{\mathbf{x}^k}\left(R+U\circ\theta_R<\infty\right)&=1-\mathbf{P}_{\mathbf{x}^k}\left(R+U\circ\theta_R=\infty\right)\\&\leq 1-\prod_{i=1}^{k}\mathbf{P}_{x^i}(\zeta<T)\\&\leq 1-\gamma^k\end{aligned}$$

for any $k\geq 1$ and $\mathbf{x}^k\in X^{(k)}$, it holds that

$$\mathbf{P}_x\left(U_2<\infty\right)\leq 1-\gamma^k.$$

Since

$$\begin{aligned}\mathbf{P}_x(U_n<\infty)&=\mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_{U_{n-1}}}\left(R+U\circ\theta_R<\infty\right);U_{n-1}<\infty\right]\\&\leq\left(1-\gamma^k\right)\mathbf{P}_x(U_{n-1}<\infty)\leq\left(1-\gamma^k\right)^{n-1}\end{aligned}$$

by induction, we get $\lim_{n\to\infty}\mathbf{P}_x(U_n<\infty)=0$, thereby completing the proof. □

We next discuss the local extinction of $\overline{\mathbf{M}}$. Let A be a relatively compact open set in X and denote by ρ_A the last exit time of $\mathbf{M}$ from A, $\rho_A=\sup\{t>0:X_t\in A\}$. Recall that $u_e^A(x)=\mathbf{P}_x\left(L_A<\infty\right)$, where $L_A=\sup\{t>0:Z_t(A)>0\}$. We then have

Proposition 2.10. *For every relatively compact open set A in X, the function u_e^A is a solution to*

$$u(x)=E_x\left[\exp\left(-A_\zeta^\mu\right);\rho_A<\infty\right]+E_x\left[\int_0^\zeta\exp\left(-A_t^\mu\right)F(u)(X_t)\,dA_t^\mu\right],\quad 0\leq u\leq 1.\qquad(2.11)$$

Proof. Let A be a relatively compact open set in X. Then the strong Markov property of $\overline{\mathbf{M}}$ implies that

$$\begin{aligned}u_e^A(x)&=\mathbf{P}_x\left(L_A<\infty\right)\\&=\mathbf{P}_x\left(L_A<\infty,\ \zeta\leq T\right)+\mathbf{P}_x\left(L_A<\infty,\ T<\zeta\right)\\&=\mathbf{P}_x\left(L_A<\infty,\ \zeta\leq T\right)+\mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_T}\left(L_A<\infty\right);T<\zeta\right].\end{aligned}$$

Since

$$\mathbf{P}_x\left(L_A<\infty,\ \zeta\leq T\right)=E_x\left[\exp\left(-A_\zeta^\mu\right);\rho_A<\infty\right]$$

and

$$\mathbf{E}_x\left[\mathbf{P}_{\mathbf{X}_T}\left(L_A<\infty\right);T<\zeta\right]=E_x\left[\int_0^\zeta\exp\left(-A_t^\mu\right)F\left(u_e^A\right)(X_t)\,dA_t^\mu\right],$$

the function u_e^A is a solution to (2.11). □

Theorem 2.11. *Assume that, for all relatively compact open set A in X, $P_x(\rho_A<\infty)=1$ for all $x\in X$, and that the branching rate μ belongs to $\mathcal{S}_\infty$ and its support is compact. Then under Assumption 1.6, the branching process $\overline{\mathbf{M}}$ extincts locally if and only if $\check{\lambda}(\mu,Q)\geq 1$.*

Proof. First suppose that $\check{\lambda}(\mu, Q) \geq 1$. We then see in a similar way to Theorem 2.4, that the constant function $u \equiv 1$ on X is a unique solution to (2.11), which implies that $u_e^A \equiv 1$ on X for each relatively compact open set A in X. Hence $\overline{\mathbf{M}}$ extincts locally. Next suppose that $\check{\lambda}(\mu, Q) < 1$. Let O be a relatively compact open set in X such that O includes the support of μ and $\check{\lambda}(\mu, Q; O) < 1$. Then $\mu|_O$ belongs to $\mathcal{S}_\infty^O$ because the support of μ is compact in O. Thus $\overline{\mathbf{M}}^O = \left(\mathbf{P}_{\mathbf{X}}^O\right)$ does not extinct by Lemma 2.3 or Theorem 2.4. In other words, $\mathbf{P}_x^O(e_0 = \infty) > 0$ for some $x \in O$. Since

$$\mathbf{P}_x^O(e_0 = \infty) \leq \mathbf{P}_x(L_O = \infty), \quad x \in O,$$

the branching process $\overline{\mathbf{M}}$ does not extinct locally. $\square$

Remark 2.12. Even if the support of the branching rate is non-compact, Theorem 2.11 remains true for a branching symmetric α-stable process. More precisely, let D be an open set in $\mathbb{R}^d$ and take an absorbing symmetric α-stable process on D as motion component. Assume that the branching rate $\mu \in \mathcal{S}_\infty^D$ satisfies $\check{\lambda}(\mu, Q) < 1$. We can then take a bounded $C^{1,1}$ domain O in D so that $\check{\lambda}(\mu, Q; O) < 1$. Let $\delta_O(x) = d(x, \partial O)$ be the Euclidian distance between x and ∂O. Then there exists a constant $C = C(O, \alpha) > 1$ such that

$$\frac{G^O(x,y)G^O(y,z)}{G^O(x,z)} \leq C\left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|y-z|^{d-\alpha}}\right), \quad x, y, z \in O$$

by [18, Theorem 1.6] and there exist positive constants C_1 and C_2 such that $C_2 > C_1 > 0$ and

$$\frac{C_1}{|x-y|^{d-\alpha}} \leq G^D(x,y) \leq \frac{C_2}{|x-y|^{d-\alpha}}, \quad x, y \in O,$$

which imply that $\mu|_O \in \mathcal{S}_\infty^O$. Hence the argument in the proof of Theorem 2.11 works.

Remark 2.13. Extinction of a branching symmetric Hunt process implies local extinction. Moreover, if $P_x(\zeta < \infty) = 1$ for all $x \in X$, then extinction and local extinction are equivalent.

We proved Theorems 2.4 and 2.11 under the assumptions that the branching rate μ belongs to the class $\mathcal{S}_\infty$ and that Assumption 1.6 holds. However, it seems so hard in general to check Assumption 1.6 for general symmetric Hunt processes. Here we give a sufficient condition for extinction or local extinction of $\overline{\mathbf{M}}$ that does not require Assumption 1.6.

Theorem 2.14. *Assume that $P_x(\zeta < \infty) = 1$ for all $x \in X$. If $\check{\lambda}(\mu, Q) > 1$, then $\overline{\mathbf{M}}$ extincts.*

Proof. Assume that $\check{\lambda}(\mu, Q) > 1$ and that $P_x(\zeta < \infty) = 1$ for all $x \in X$. Let u be a solution to (2.1), and let $v = 1 - u$. We can then show that

$$0 \leq v(x) < E_x\left[\int_0^\zeta \exp\left(-A_t^\mu\right) v(X_t)\, dA_t^{Q\mu}\right] \tag{2.12}$$

in a way similar to that yielding (2.5). Since

$$0 \leq v(x) < E_x\left[\int_0^\zeta \exp\left(-A_t^\mu\right) \frac{(A_t^{Q\mu})^n}{n!}\, dA_t^{Q\mu}\right], \quad n \geq 1$$

by the iterations of the inequality (2.12), it holds that

$$\sum_{n=0}^{\infty} v(x) \leq E_x \left[\int_0^{\zeta} \exp\left(A_t^{(Q-1)\mu} \right) dA_t^{Q\mu} \right].$$

Noting that $\check{\lambda}(\mu; Q) > 1$ if and only if the right hand side above is bounded by Theorem 1.2, we obtain $v \equiv 0$ and $u \equiv 1$ on X, which implies that $u_e \equiv 1$ on X. $\square$

We can obtain the following in a similar way to that yielding Theorem 2.14.

Theorem 2.15. *Assume that, for every relatively compact open set A in X, $P_x(\rho_A < \infty) = 1$ for all $x \in X$. If $\check{\lambda}(\mu, Q) > 1$, then $\overline{\mathbf{M}}$ extincts locally.*

2.2 Examples

2.2.1 Branching Brownian motions

Example 2.16. Suppose that $d = 1$. Let $\mathbf{M}^{\nu}, \nu \in \mathcal{S}_1$, be the killed Brownian motion with respect to $\exp(-A_t^{\nu})$ and let $\overline{\mathbf{M}^{\nu}}$ be the branching Brownian motion with motion component $\mathbf{M}^{\nu}$ and branching rate $\mu \in \mathcal{K}_{\infty}^{\mathbb{R}}$. First take $\mu = \delta_0$ and $\nu = \delta_{-a} + \delta_a$ for $a > 0$. Since

$$\inf \left\{ \frac{1}{2}\mathbf{D}(u,u) + u(-a)^2 + u(a)^2 : u \in C_0^{\infty}(\mathbb{R}),\ u(0)^2 = 1 \right\} = \frac{2}{1+2a}$$

by Example 6.2 below, the branching process $\overline{\mathbf{M}^{\nu}}$ extincts if and only if

$$Q(0) \leq 1 + \frac{2}{1+2a}.$$

Next take $\mu = \delta_{-b} + \delta_b$ for $b > 0$ and $\nu = \delta_0$. Suppose that $Q(b) = Q(-b) = Q$. Since

$$\inf \left\{ \frac{1}{2}\mathbf{D}(u,u) + u(0)^2 : u \in C_0^{\infty}(\mathbb{R}),\ u(-b)^2 + u(b)^2 = 1 \right\} = \frac{1}{2(1+b)}$$

by Example 6.2, the branching process $\overline{\mathbf{M}^{\nu}}$ extincts if and only if

$$Q \leq 1 + \frac{1}{2(1+b)}.$$

Example 2.17. Let M be a spherically symmetric Riemannian manifold with a pole o and consider the Brownian motion on M. Denote by $(\mathcal{E}, \mathcal{F})$ the associated Dirichlet form on $L^2(M; V)$:

$$\mathcal{E}(u,u) = \frac{1}{2} \int_M |\nabla u|^2 \, dV,$$
$$\mathcal{F} = \text{the closure of } C_0^{\infty}(M) \text{ with respect to } \mathcal{E}(\cdot,\cdot) + \|\cdot\|_{L^2(M;V)}^2,$$

where V is the Riemannian volume of M. Let $B(r) = \{x \in M : d(x,o) < r\}$ and $\partial B(r) = \{x \in M : d(x,o) = r\}$, where d is the Riemannian distance of M. Denote by δ_r the surface measure of $\partial B(r)$ and let

$$S(r) = \delta_r(\partial B(r)) \quad \text{and} \quad G(r) = \int_r^{\infty} \frac{1}{S(\rho)} \, d\rho.$$

We now set

$$\check{\lambda}(\delta_R; M \setminus B(r)) = \inf\left\{\mathcal{E}(u,u) : u \in C_0^\infty(M \setminus B(r)),\ \int_{\partial B(R)} u^2\, d\delta_R = 1\right\}$$

for r and R with $R > r > 0$. Then the following results are shown by Takeda [56]; if $(Q-1)S(R)G(R) > 1/2$, then

$$\check{\lambda}(\delta_R; M \setminus B(r)) \geq Q - 1 \iff r_0 \leq r < R,$$

where the positive constant r_0 is a unique root of

$$G(r) = \frac{2(Q-1)S(R)G(R)^2}{2(Q-1)S(R)G(R) - 1}.$$

On the other hand, if $(Q-1)S(R)G(R) \leq 1/2$, then

$$\check{\lambda}(\delta_R; M \setminus B(r)) > Q - 1$$

for any $r < R$.

Let us denote by $\mathbf{M}$ the Brownian motion on M and by $\mathbf{M}^r$ the absorbing Brownian motion on $M \setminus B(r)$. Let $\overline{\mathbf{M}^r}$ be the branching Brownian motion with motion component $\mathbf{M}^r$, branching rate δ_R and branching mechanism $\{p_n(x)\}_{n=0}^\infty$ such that $Q(x) = \sum_{n=0}^\infty n p_n(x) \equiv Q$. Theorem 2.11 and Remark 6.6 then imply the following; if $(Q-1)S(R)G(R) > 1/2$, then $\overline{\mathbf{M}^r}$ extincts locally if and only if $r_0 \leq r < R$. On the other hand, if $(Q-1)S(R)G(R) \leq 1/2$, then $\overline{\mathbf{M}^r}$ extincts locally for any $r < R$. For instance, take the d-dimensional hyperbolic space $\mathbb{H}^d$ as M (see Example 3.3 of [32] for definition).

(i) For $d = 2$, $S(R)G(R)$ is strictly increasing,

$$\lim_{R \downarrow 0} S(R)G(R) = 0 \quad \text{and} \quad \lim_{R \to \infty} S(R)G(R) = 1$$

([52, Example 2.6]). Hence if $Q > 3/2$, then there exists a unique root R_0 such that $(Q-1)S(R_0)G(R_0) = 1/2$. Moreover, if $R > R_0$, then $\overline{\mathbf{M}^r}$ extincts locally if and only if $r_0 \leq r < R$. If $R \leq R_0$, then $\overline{\mathbf{M}^r}$ extincts locally for any $r < R$. On the other hand, if $Q \leq 3/2$, then $(Q-1)S(R)G(R) < 1/2$ for all $R > 0$, and consequently $\overline{\mathbf{M}^r}$ extincts locally for any $r < R$.

(ii) For $d = 3$,

$$(Q-1)S(R)G(R) > 1/2 \iff Q > 2 + \frac{1}{e^{2R} - 1} \tag{2.13}$$

by Example 2.6 of [52]. Hence, if Q satisfies the right hand side of (2.13), then $\overline{\mathbf{M}^r}$ extincts locally if and only if $r_0 \leq r < R$. Otherwise, $\overline{\mathbf{M}^r}$ extincts locally for any $r < R$.

(iii) For $d \geq 4$, $S(R)G(R) < 1/(d-1)$ by Example 2.6 of [52]. Therefore, if $Q \leq (d+1)/2$, then $\overline{\mathbf{M}^r}$ extincts locally for any $r < R$.

More detailed properties are studied for branching Brownian motions on $\mathbb{H}^2$ in [39], and for branching Markov processes on $\mathbb{H}^d$ in [36].

2.2.2 Branching symmetric α-stable processes

Let $\mathbf{M}^\alpha = (X_t, P_x)$ be the symmetric α-stable process on $\mathbb{R}^d$. Let $\mathbf{M}^D$ be the absorbing α-stable process on an open set D in $\mathbb{R}^d$ and $(\mathcal{E}^D, \mathcal{F}^D)$ the associated Dirichlet form on $L^2(D)$. Define

$$\check{\lambda}(\mu; D) = \inf\left\{\mathcal{E}^D(u,u) : u \in C_0^\infty(D), \int_D u^2\, d\mu = 1\right\}$$

for $\mu \in \mathcal{K}_\infty^D$.

Example 2.18. Suppose that $d = 1$ and $1 < \alpha \leq 2$. Then $\mathbf{M}^\alpha$ is recurrent and one point has positive capacity. Let $D = \mathbb{R} \setminus \{0\}$, $\mu = \delta_a$, $a > 0$, and $p_0(x) + p_2(x) \equiv 1$ on D. Then $Q(x) = 2p_2(x)$. Since

$$\lambda(\delta_a; \mathbb{R}^d \setminus \{0\}) = -\frac{\Gamma(\alpha)\cos\left(\dfrac{\pi\alpha}{2}\right)}{2a^{\alpha-1}}$$

by Example 6.6 below, we see from Theorem 2.4 and Theorem 2.8 that

$$Q(a) < 1 - \frac{\Gamma(\alpha)\cos\left(\dfrac{\pi\alpha}{2}\right)}{2a^{\alpha-1}} \Rightarrow \mathbf{P}_x(e_0 < \infty) \equiv 1, \sup_{x \in \mathbb{R}\setminus\{0\}} \mathbf{E}_x\left[N_{\{0\}}\right] < \infty$$

$$Q(a) = 1 - \frac{\Gamma(\alpha)\cos\left(\dfrac{\pi\alpha}{2}\right)}{2a^{\alpha-1}} \Rightarrow \mathbf{P}_x(e_0 < \infty) \equiv 1, \sup_{x \in \mathbb{R}\setminus\{0\}} \mathbf{E}_x\left[N_{\{0\}}\right] = \infty$$

$$Q(a) > 1 - \frac{\Gamma(\alpha)\cos\left(\dfrac{\pi\alpha}{2}\right)}{2a^{\alpha-1}} \Rightarrow \mathbf{P}_x(e_0 < \infty) < 1, \sup_{x \in \mathbb{R}\setminus\{0\}} \mathbf{E}_x\left[N_{\{0\}}\right] = \infty.$$

In particular, if

$$0 < a \leq \left\{-\frac{\Gamma(\alpha)\cos\left(\dfrac{\pi\alpha}{2}\right)}{2}\right\}^{1/(\alpha-1)},$$

then this branching symmetric α-stable process extincts even if $p_2(a) = 1$.

Example 2.19. Suppose that $1 < \alpha \leq 2$ and $d > \alpha$. Then $\mathbf{M}^\alpha$ is transient and the surface of a sphere has positive capacity. Let δ_r be the surface measure of a sphere $\partial B(r) = \{x \in \mathbb{R}^d : |x| = r\}$. Take $\mu = \delta_r$ and assume that $Q(x) \equiv Q$. Using Example 4.1 of [58], we see from Theorem 2.11 and Remark 6.6 that, if $Q > 1$, then $\overline{\mathbf{M}^\alpha}$ extincts locally if and only if

$$0 < r \leq \left\{\frac{\sqrt{\pi}\Gamma\left(\dfrac{d+\alpha}{2} - 1\right)\Gamma\left(\dfrac{\alpha}{2}\right)}{(Q-1)\Gamma\left(\dfrac{\alpha-1}{2}\right)\Gamma\left(\dfrac{d-\alpha}{2}\right)}\right\}^{1/(\alpha-1)}. \tag{2.14}$$

On the other hand, if $Q \leq 1$, then $\overline{\mathbf{M}^\alpha}$ extincts locally for any $r > 0$.

Let $\overline{\delta_r}$ be the normalized surface measure of $\partial B(r)$, $\overline{\delta_r} = \delta_r/\delta_r(\partial B(r))$. Take $\mu = \overline{\delta_r}$ and assume that $Q(x) \equiv Q$. Noting that $\lambda\left(\overline{\delta_r}; \mathbb{R}^d\right) = \delta_r(\partial B(r))\lambda(\delta_r; \mathbb{R}^d)$ and $\delta_r(\partial B(r)) = 2\pi^{d/2}r^{d-1}/\Gamma(d/2)$. we see that if $Q > 1$, then $\overline{\mathbf{M}^\alpha}$ extincts locally if and only if

$$r \geq \left\{\frac{(Q-1)\Gamma\left(\dfrac{d}{2}\right)\Gamma\left(\dfrac{\alpha-1}{2}\right)\Gamma\left(\dfrac{d-\alpha}{2}\right)}{2\pi^{(d+1)/2}\Gamma\left(\dfrac{\alpha}{2}\right)\Gamma\left(\dfrac{d+\alpha}{2} - 1\right)}\right\}^{1/(d-\alpha)}. \tag{2.15}$$

On the other hand, if $Q \le 1$, then $\overline{\mathbf{M}^{\alpha}}$ extincts locally for any $r > 0$.

Example 2.20. Suppose that $0 < \alpha \le 2$ and $d > \alpha$. Let $\mathbf{1}_{B(r)}\, dx$ be the d-dimensional Lebesgue measure restricted on a ball $B(r) = \{x \in \mathbb{R}^d : |x| < r\}$. Take $\mu(dx) = \mathbf{1}_{B(r)}\, dx$ and assume that $Q(x) \equiv Q$. If $Q > 1$ and

$$0 < r \le \left\{ \frac{2^{\alpha-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)}{(Q-1)\Gamma\left(\frac{d-\alpha}{2}\right)} \right\}^{1/\alpha}, \tag{2.16}$$

then $\overline{\mathbf{M}^{\alpha}}$ extincts locally by Example 6.4 below. On the other hand, if $Q \le 1$, then $\overline{\mathbf{M}^{\alpha}}$ extincts locally for any $r > 0$.

Chapter 3

Exponential growth of the numbers of particles for branching symmetric Markov processes

We study the exponential growth of the numbers of particles for branching symmetric Hunt processes by the principal eigenvalues of associated Schrödinger operators under the assumption that the Schrödinger operators have spectral gaps.

3.1 Exponential growth of the numbers of particles

Let X be a locally compact separable metric space and m a positive smooth Radon measure on X with full support. Let $\mathbf{M} = (X_t, P_x)$ be an m-symmetric Hunt process on X. Throughout this section, we assume that $\mathbf{M}$ is transient and satisfies Assumption 1.3. Let $\overline{\mathbf{M}} = (\mathbf{X_t}, \mathbf{P}_x, \mathcal{G}_t)$ be the branching symmetric Hunt process with motion component $\mathbf{M}$, branching rate $\mu \in \mathcal{K}_\infty$ and branching mechanism $\{p_n(x)\}_{n\geq 0}$. Recall that $Q(x) = \sum_{n=0}^{\infty} np_n(x)$ and suppose that $\sup_{x\in X} Q(x) < \infty$.

We proved in Lemma 2.9 that, if $P_x(\zeta < \infty) = 1$ for any $x \in X$, then

$$\{e_0 = \infty\} = \left\{\lim_{t\to\infty} Z_t = \infty\right\} \quad \mathbf{P}_x\text{-a.s.}$$

for any $x \in X$. This result says that, if the branching process $\overline{\mathbf{M}}$ does not extinct, then

$$\mathbf{P}_x\left(\lim_{t\to\infty} Z_t = \infty \,\middle|\, e_0 = \infty\right) = 1.$$

We first study the exponential growth of Z_t in terms of the principal eigenvalue

$$\lambda_1((Q-1)\mu) = \inf\left\{\mathcal{E}(u,u) - \int_X u^2(Q-1)\,d\mu : u \in \mathcal{F}, \int_X u^2\,dm = 1\right\}. \tag{3.1}$$

From now on, we suppose that $\lambda_1 := \lambda_1((Q-1)\mu) < 0$. We denote by h the ground state corresponding to λ_1. Then

$$h(x) = e^{\lambda_1 t} E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) h(X_t); t < \zeta\right]. \tag{3.2}$$

Define

$$M_t = e^{\lambda t} \sum_{i=1}^{Z_t} h(\mathbf{X}_t^i), \quad t \geq 0. \tag{3.3}$$

Then M_t is a $\mathbf{P}_x$-martingale by (1.19) and (3.2), and thus there exists a limit $M_\infty := \lim_{t\to\infty} M_t \in [0,\infty)$ $\mathbf{P}_x$-a.s. Furthermore, it follows from (1.20) and (3.2) that

$$\begin{aligned} \mathbf{E}_x\left[M_t^2\right] &= e^{2\lambda_1 t} E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) h(X_t)^2; t<\zeta\right] \\ &\quad + E_x\left[\int_0^{t\wedge\zeta} \exp\left(2\lambda_1 s + A_s^{(Q-1)\mu}\right) h(X_s)^2\, dA_s^{R\mu}\right], \end{aligned} \tag{3.4}$$

where $R(x) = \sum_{n=0}^\infty n(n-1)p_n(x)$.

Lemma 3.1. *Assume that* $\sup_{x\in X} R(x) < \infty$. *Then* M_t *is square integrable.*

Proof. Since

$$\begin{aligned} e^{2\lambda_1 t} E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) h(X_t)^2; t<\zeta\right] &\leq e^{\lambda_1 t}\|h\|_\infty e^{\lambda_1 t} E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) h(X_t); t<\zeta\right] \\ &= e^{\lambda_1 t}\|h\|_\infty h(x) \end{aligned}$$

by (3.2), the last term above converges to 0 as $t\to\infty$. Hence it follows from (3.4) that

$$\begin{aligned} \lim_{t\to\infty} \mathbf{E}_x\left[M_t^2\right] &= E_x\left[\int_0^{\zeta} \exp\left(2\lambda_1 s + A_s^{(Q-1)\mu}\right) h(X_s)^2\, dA_s^{R\mu}\right] \\ &\leq \|h\|_\infty^2 \|R\|_\infty \sup_{x\in X} E_x\left[\int_0^{\zeta} \exp\left(2\lambda_1 s + A_s^{(Q-1)\mu}\right)\, dA_s^{\mu}\right]. \end{aligned} \tag{3.5}$$

Since

$$\inf\left\{\mathcal{E}(u,u) - \int_X u^2\,(Q-1)d\mu - 2\lambda_1\int_X u^2\,dm : u\in\mathcal{F},\ \int_X u^2\,dm = 1\right\} = -\lambda_1 > 0$$

by the definition of λ_1, Lemma 3.5 of [52] shows that

$$\inf\left\{\mathcal{E}(u,u) + \int_X u^2\,d\mu - 2\lambda_1\int_X u^2\,dm : u\in\mathcal{F},\ \int_X u^2\,Qd\mu = 1\right\} > 1.$$

Then the last term of (3.5) is finite by Theorem 1.2, whence M_t is square integrable. □

Lemma 3.1 tells us that $\mathbf{E}_x\left[M_\infty\right] = h(x) > 0$, which yields that $\mathbf{P}_x\left(M_\infty \in (0,\infty)\right) > 0$ for any $x\in X$. It also holds that

$$\mathbf{E}_x\left[M_\infty^2\right] = E_x\left[\int_0^{\zeta} \exp\left(2\lambda_1 s + A_s^{(Q-1)\mu}\right) h(X_s)^2\, dA_s^{R\mu}\right].$$

Recall that the extinction probability u_e is a minimal solution to (2.1) by Proposition 2.1. We then obtain

Lemma 3.2. *Suppose that* $P_x(\zeta<\infty) = 1$ *for any* $x\in X$. *If* $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy) < \infty$, *then the equation* (2.1) *has just two solutions,* $u\equiv 1$ *and* u_e.

Proof. Let u be a solution to (2.1) such that $u(x_0) < 1$ for $x_0 \in X$. Since u is finely continuous by Lemma 2.2, it follows from (2.1) that $P_{x_0}(\sigma_{O\cap F^\mu} < \infty) > 0$, where $O = \{x \in X : u(x) < 1\}$ and F^μ is the fine support of the measure μ defined in (1.8). Moreover, by the irreducibility of $\mathbf{M}$, it holds that $P_x(\sigma_{O\cap F^\mu} < \infty) > 0$ for any $x \in X$, which implies that $u < 1$ on X.

As a direct calculation yields that

$$E_x\left[\exp\left(-A^\mu_\zeta\right)\right] = 1 - E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right)\, dA^\mu_t\right],$$

the equation (2.1) is equivalent to that

$$v = G^\mu\left(\left(F(1) - F(1-v)\right)\mu\right) \text{ on } X,$$

where $v = 1 - u > 0$. Since the function $v_e = 1 - u_e > 0$ is a solution to the equation above, we see that

$$\begin{aligned}\int_X v(F(1) - F(1-v_e)))\, d\mu &= \int_X G^\mu\left(\left(F(1) - F(1-v)\right)\mu\right)\left(F(1) - F(1-v_e)\right) d\mu \\ &= \int_X G^\mu\left(\left(F(1) - F(1-v_e)\right)\mu\right)\left(F(1) - F(1-v)\right) d\mu \\ &= \int_X v_e(F(1) - F(1-v))\, d\mu.\end{aligned}$$

Here the integrability of the terms above follows by the assumption on μ and the second equality holds by Theorem 3.2 (iv) of [2]. Since $F(\cdot)$ is strictly convex and $v_e \geq v > 0$, it holds that

$$\frac{F(1) - F(1-v)}{1 - (1-v)} = \frac{F(1) - F(1-v_e)}{1 - (1-v_e)} \quad \mu\text{-a.e.},$$

which shows that $u = u_e$ μ-a.e. Using (2.1), we have $u = u_e$ on X. □

Proposition 3.3. *Suppose that $P_x(\zeta < \infty) = 1$ for any $x \in X$. If $\sup_{x\in X} R(x) < \infty$ and $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy) < \infty$, then*

$$\{e_0 = \infty\} = \{M_\infty > 0\} \quad \mathbf{P}_x\text{-a.s.}$$

for any $x \in X$.

Proof. Since $\lambda_1 < 0$ and

$$M_t \leq e^{\lambda_1 t} Z_t \|h\|_\infty, \tag{3.6}$$

it holds that

$$\{M_\infty > 0\} \subset \{e_0 = \infty\}.$$

By the assumption on the lifetime,

$$\mathbf{P}_x(T = \infty, e_0 = \infty) = E_x\left[\exp\left(-A^\mu_\zeta\right); \zeta = \infty\right] = 0.$$

Noting that

$$\{T = \infty\} \subset \{e_0 < \infty\} \subset \{M_\infty = 0\},$$

we see that

$$\begin{aligned}
\mathbf{P}_x(M_\infty = 0) &= \mathbf{P}_x(M_\infty = 0, T = \infty) + \mathbf{P}_x(M_\infty = 0, T < \infty) \\
&= \mathbf{P}_x(T = \infty) + \mathbf{P}_x(M_\infty = 0, T < \infty) \\
&= E_x\left[\exp\left(-A^\mu_\zeta\right); \zeta < \infty\right] + E_x\left[\int_0^\zeta \exp\left(-A^\mu_t\right) F(\mathbf{P}_\cdot(M_\infty = 0))(X_t)\, dA^\mu_t\right],
\end{aligned}$$

that is, the function $\mathbf{P}_x(M_\infty = 0)$ is a solution to (2.1). Since $\mathbf{P}_x\left(M_\infty = 0\right) < 1$, it follows from Lemma 3.2 that $\mathbf{P}_x(M_\infty = 0) = u_e(x)$ for any $x \in X$. Namely, $\mathbf{P}_x(M_\infty > 0) = \mathbf{P}_x(e_0 = \infty)$ for any $x \in X$, which completes the proof. $\square$

Theorem 3.4. *Suppose that $P_x\left(\zeta < \infty\right) = 1$ for any $x \in X$.*
(i) *If $\sup_{x\in X} R(x) < \infty$ and $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy) < \infty$, then*

$$\mathbf{P}_x\left(M_\infty \in (0,\infty)\,|\, e_0 = \infty\right) = 1, \quad x \in X. \tag{3.7}$$

As a consequence,

$$\mathbf{P}_x\left(\liminf_{t\to\infty} e^{\lambda_1 t} Z_t > 0 \,\middle|\, e_0 = \infty\right) = 1, \quad x \in X. \tag{3.8}$$

(ii) *If $\sup_{x\in X} R(x) < \infty$ and $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy) < \infty$, then for any $\kappa > \lambda_1$,*

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t = \infty \,\middle|\, e_0 = \infty\right) = 1, \quad x \in X. \tag{3.9}$$

(iii) *For any $\kappa < \lambda_1$,*

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} \sum_{i=1}^{Z_t} h(\mathbf{X}^i_t) = 0\right) = 1, \quad x \in X \tag{3.10}$$

and

$$\mathbf{P}_x\left(\liminf_{t\to\infty} e^{\kappa t} Z_t = 0\right) = 1, \quad x \in X. \tag{3.11}$$

Furthermore, if X is Green bounded for $\mathbf{M}$, that is, if $\sup_{x\in X} E_x\left[\zeta\right] < \infty$, then, for any $\kappa < \lambda_1$,

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t = 0\right) = 1, \quad x \in X. \tag{3.12}$$

Proof. The equation (3.7) follows from Proposition 3.3. Since

$$\{M_\infty > 0\} \subset \left\{\liminf_{t\to\infty} e^{\lambda_1 t} Z_t > 0\right\} \subset \left\{\lim_{t\to\infty} e^{\kappa t} Z_t = \infty\right\}$$

for $\kappa > \lambda$ by (3.6), we have (3.8) and (3.9).

Suppose that $\kappa < \lambda$. Then the equation (3.10) holds by Lemma 3.1. By (1.19),

$$\begin{aligned}
e^{\kappa t}\mathbf{E}_x\left[Z_t\right] &= E_x\left[\exp\left(\kappa t + A^{(Q-1)\mu}_t\right); t < \zeta\right] \\
&= e^{\kappa t} E_x\left[\exp\left(-A^\mu_t\right) \int_0^t \exp\left(A^{Q\mu}_s\right)\, dA^{Q\mu}_s; t < \zeta\right] + e^{\kappa t} E_x\left[\exp\left(-A^\mu_t\right); t < \zeta\right].
\end{aligned} \tag{3.13}$$

Choose a positive constant ε such that $0 < \varepsilon < \lambda_1 - \kappa$. Then the last term above is not greater than

$$e^{(\kappa-\lambda_1+\varepsilon)t} E_x\left[\int_0^\zeta \exp\left((\lambda_1 - \varepsilon)s + A^{(Q-1)\mu}_s\right)\, dA^{Q\mu}_s\right] + e^{\kappa t} E_x\left[\exp\left(-A^\mu_t\right); t < \zeta\right]. \tag{3.14}$$

By the same argument as in Lemma 3.1, it follows that

$$\sup_{x\in X} E_x\left[\int_0^{\zeta} \exp\left((\lambda_1-\varepsilon)s + A_s^{(Q-1)\mu}\right) dA_s^{Q\mu}\right] < \infty,$$

and thus the term of (3.14) converges to 0 as $t \to \infty$. Hence by Fatou's lemma,

$$\mathbf{E}_x\left[\liminf_{t\to\infty} e^{\kappa t} Z_t\right] \le \lim_{t\to\infty} e^{\kappa t}\mathbf{E}_x\left[Z_t\right] = 0,$$

which implies (3.11).

From now on, we assume that X is Green bounded for $\mathbf{M}$. Let

$$u_\kappa(x) = E_x\left[\exp\left(\kappa\zeta + A_\zeta^{(Q-1)\mu}\right)\right].$$

Then $\sup_{x\in X} u_\kappa(x) < \infty$ by Theorem 5.2 of [14] or Theorem 2.4 of [52]. Moreover, Jensen's inequality yields that

$$\inf_{x\in X} u_\kappa(x) \ge \exp\left(\kappa \sup_{x\in X} E_x\left[\zeta\right] - \sup_{x\in X} E_x\left[A_\zeta^\mu\right]\right) > 0,$$

where we note that $\sup_{x\in X} E_x\left[A_\zeta^\mu\right] < \infty$ by (1.10). By the definition of u_κ and (1.19),

$$\begin{aligned} e^{\kappa t}\mathbf{E}_x\left[\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i)\right] &= e^{\kappa t}E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) u_\kappa(X_t); t<\zeta\right] \\ &= e^{\kappa t}E_x\left[\exp\left(A_t^{(Q-1)\mu}\right) E_{X_t}\left[\exp\left(\kappa\zeta + A_\zeta^{(Q-1)\mu}\right)\right]; t<\zeta\right]. \end{aligned}$$

Then the last term above is equal to

$$E_x\left[\exp\left(\kappa\zeta + A_\zeta^{(Q-1)\mu}\right); t<\zeta\right] \le u_\kappa(x)$$

by the Markov property. Since $e^{\kappa t}\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i)$ is a nonnegative $\mathbf{P}_x$-supermartingale such that

$$\sup_{(x,t)\in X\times[0,\infty)} e^{\kappa t}\mathbf{E}_x\left[\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i)\right] \le \sup_{x\in X} u_\kappa(x) < \infty,$$

there exists a limit $\lim_{t\to\infty} e^{\kappa t}\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i) < \infty$ $\mathbf{P}_x$-a.s. for any $x \in X$. Moreover, we see that $\limsup_{t\to\infty} e^{\kappa t} Z_t < \infty$ $\mathbf{P}_x$-a.s. because $\inf_{x\in X} u_\kappa(x) > 0$ and

$$\left(\inf_{x\in X} u_\kappa(x)\right) e^{\kappa t} Z_t \le e^{\kappa t}\sum_{i=1}^{Z_t} u_\kappa(\mathbf{X}_t^i).$$

Noting that $\kappa < \lambda_1$ is arbitrary, we have (3.12). □

We next study the exponential growth of the number of particles in every open set. In the sequel, we assume that $\lambda_1 := \lambda_1((Q-1)\mu) < 0$. Let A be an open set in X. Note that, if $\mathbf{P}_x(L_A = \infty) > 0$ for some $x \in X$, then $\mathbf{P}_x(L_A = \infty) > 0$ for any $x \in X$ by the irreducibility of $\mathbf{M}$.

Lemma 3.5. *Assume that the support of the branching rate μ is compact. Then, for any non-empty open set A in X,*

$$\mathbf{P}_x\left(\limsup_{t\to\infty} Z_t(A)=\infty\right)>0, \quad x\in X. \tag{3.15}$$

Moreover, if $P_x(\rho_A<\infty)=1$ for any $x\in X$, then

$$\mathbf{P}_x\left(\limsup_{t\to\infty} Z_t(A)=0 \text{ or } \infty\right)=1, \quad x\in X. \tag{3.16}$$

Namely,

$$\{L_A=\infty\}=\left\{\limsup_{t\to\infty} Z_t(A)=\infty\right\} \quad \mathbf{P}_x\text{-a.s.}, \quad x\in X.$$

To prove Lemma 3.5, we consider the following equation:

$$u(x)=E_x\left[\exp\left(-A_\zeta^\mu\right)\right]+E_x\left[\int_0^\zeta \exp\left(-A_t^\mu\right)F(u)(X_t)\,dA_t^\mu\right], \quad 0\le u\le 1. \tag{3.17}$$

We can then prove the following by the same way as in Lemma 3.2.

Lemma 3.6. *Assume that $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy)<\infty$. If the functions u_1 and u_2 are solutions to* (3.17) *respectively, and $u_1\le u_2<1$ on X, then $u_1=u_2$ on X.*

Proof of Lemma 3.5. Let O be a relatively compact open set in X such that O includes the support of μ and $\lambda_1<\lambda_1(\mu,Q;O)<0$, where

$$\lambda_1(\mu,Q;O)=\inf\left\{\mathcal{E}^O(u,u)-\int_O u^2\,(Q-1)d\mu : u\in\mathcal{F}^O, \int_O u^2\,dm=1\right\}.$$

Since the measure $\mu|_O$ belongs to $\mathcal{S}_\infty^O$, the branching process $\overline{\mathbf{M}^O}=(\mathbf{P}_{\mathbf{x}}^O)$ does not extinct by Theorem 2.4, and thus,

$$\mathbf{P}_x\left(\lim_{t\to\infty} Z_t(O)=\infty\right)\ge \mathbf{P}_x^O\left(\lim_{t\to\infty} Z_t=\infty\right)>0, \quad x\in O.$$

Furthermore, the left hand side above is positive for any $x\in X$ by the irreducibility of $\mathbf{M}$.

Let us denote by $p_t^{(Q-1)\mu}(x,y)$ the integral kernel of the Feynman-Kac semigroup $p_t^{(Q-1)\mu}$ as defined in (1.14). Then $p_t^{(Q-1)\mu}(x,A):=\int_A p_t^{(Q-1)\mu}(x,y)\,m(dy)$ is bounded and continuous on X by Theorem 1.4 (i) and

$$p:=\inf_{x\in O} p_1^{(Q-1)\mu}(x,A)>0$$

by the irreducibility of $\mathbf{M}$. Since

$$\mathbf{E}_x\left[Z_t(A)\right]=E_x\left[\exp\left(A_t^{(Q-1)\mu}\right); t<\zeta, X_t\in A\right]$$

by (1.19), it holds that

$$\inf_{x\in O}\mathbf{E}_x\left[Z_1(A)\right]=p>0,$$

and thus

$$\inf_{x\in O}\mathbf{P}_x(Z_1(A)\ge 1)>0. \tag{3.18}$$

Let q be a nonnegative constant such that

$$e^{-q} = \sup_{x \in O} \mathbf{E}_x \left[\exp\left(-Z_1(A)\right) \right].$$

Then it holds that $0 < q \le p$ because the right hand side above is less than one by (3.18) and

$$\sup_{x \in O} \mathbf{E}_x \left[\exp\left(-Z_1(A)\right) \right] \ge \exp\left(- \inf_{x \in O} \mathbf{E}_x \left[Z_1(A) \right] \right)$$

by Jensen's inequality. Choose a positive constant $\overline{q}$ such that $0 < \overline{q} < q$. Then for any $\mathbf{x}^n = (x^1, x^2, x^3, \cdots x^n) \in O^{(n)}$,

$$\begin{aligned} \mathbf{P}_{\mathbf{x}^n}(Z_1(A) < \overline{q} Z_0(O)) &= \mathbf{P}_{\mathbf{x}^n}\left(\exp\left(-Z_1(A)\right) > \exp\left(-\overline{q} Z_0(O)\right) \right) \\ &\le e^{n\overline{q}} \prod_{i=1}^{n} \mathbf{E}_{x^i} \left[\exp\left(-Z_1(A)\right) \right] \end{aligned}$$

by Chebyshev's inequality. Since the last term above is not greater than $e^{(\overline{q}-q)} < 1$ for any $n \ge 1$ by the definition of q, it holds that

$$\sup_{n \ge 1,\, \mathbf{x}^n \in O^{(n)}} \mathbf{P}_{\mathbf{x}^n}(Z_1(A) < \overline{q} Z_0(O)) < 1.$$

Namely,

$$\inf_{n \ge 1,\, \mathbf{x}^n \in O^{(n)}} \mathbf{P}_{\mathbf{x}^n}(Z_1(A) \ge \overline{q} Z_0(O)) > 0.$$

Let us define

$$A_m = \{ Z_m(A) \ge \overline{q} Z_{m-1}(O) \}$$

for any positive integer $m \ge 1$ and

$$\Omega_0 = \left\{ \lim_{t \to \infty} Z_t(O) = \infty \right\}. \tag{3.19}$$

Then by the Markov property,

$$\begin{aligned} \mathbf{P}_x(A_{m+1} \,|\, \mathcal{G}_m)(\omega) &= \mathbf{P}_{\mathbf{X}_m(\omega)}(Z_1(A) \ge \overline{q} Z_0(O)) \\ &\ge \inf_{n \ge 1,\, \mathbf{x}^n \in O^{(n)}} \mathbf{P}_{\mathbf{x}^n}(Z_1(A) \ge \overline{q} Z_0(O)) > 0 \end{aligned}$$

for any $x \in X$ and $\omega \in \Omega_0$, and hence

$$\sum_{m=0}^{\infty} \mathbf{P}_x(A_{m+1} \,|\, \mathcal{G}_m)(\omega) = \infty.$$

Noting that

$$\left\{ \sum_{m=0}^{\infty} \mathbf{P}_x(A_{m+1} \,|\, \mathcal{G}_m) = \infty \right\} = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_m$$

by [25, p.237, Corollary 3.2], we obtain (3.15).

From now on, we assume that A is an open set in X such that $P_x(\rho_A < \infty) = 1$ for any $x \in X$. Set

$$u_1(x) = \mathbf{P}_x\left(\lim_{t \to \infty} Z_t(A) = 0 \right)$$

and

$$u_2(x) = \mathbf{P}_x\left(\limsup_{t\to\infty} Z_t(A) < \infty\right).$$

We then see in a similar way to Proposition 3.3 that the functions u_1 and u_2 are solutions to (3.17) respectively, by the assumption on A. Since it holds that $u_1 \leq u_2 < 1$ by definition, Lemma 3.6 implies that $u_1 = u_2$ on X, which leads us to (3.16). □

Proposition 3.7. *Assume that the support of the branching rate μ is compact. Then, for any non-empty open set A in X and $\kappa > \lambda_1$,*

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty\right) > 0, \quad x \in X. \tag{3.20}$$

Moreover, if $P_x(\rho_A < \infty) = 1$ for any $x \in X$, then

$$\{L_A = \infty\} = \left\{\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty\right\} \quad \mathbf{P}_x\text{-a.s.}, \quad x \in X,$$

and

$$\{L_A < \infty\} = \left\{\lim_{t\to\infty} e^{\kappa t} Z_t(A) = 0\right\} \quad \mathbf{P}_x\text{-a.s.}, \quad x \in X.$$

Proof. For any $\kappa > \lambda_1$, there exists a relatively compact open set O in X such that O includes the support of μ and $\lambda_1 < \lambda_1(\mu, Q; O) < \kappa$. Then, by Theorem 3.4 (ii),

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t(O) = \infty\right) \geq \mathbf{P}_x^O\left(\lim_{t\to\infty} e^{\kappa t} Z_t = \infty\right) > 0, \quad x \in O.$$

Moreover, the left hand side above is positive for any $x \in X$ by the irreducibility of $\mathbf{M}$. If we replace Ω_0 defined in (3.19) with

$$\left\{\lim_{t\to\infty} e^{\kappa t} Z_t(O) = \infty\right\},$$

then (3.20) follows by the same way as in Lemma 3.5.

From now on, let A be an open set in X such that $P_x(\rho_A < \infty) = 1$ for any $x \in X$. Set

$$u_1(x) = \mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t(A) = 0\right)$$

and

$$u_2(x) = \mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) < \infty\right).$$

Then it follows from (3.20) that $u_A \leq u_1 \leq u_2 < 1$ on X, where $u_A(x) = \mathbf{P}_x(L_A < \infty)$. Furthermore, by noting that u_A, u_1 and u_2 are solutions to (3.17) respectively, Lemma 3.6 implies that $u_A = u_1 = u_2$ on X, which completes the proof. □

Theorem 3.8. (i) *For any relatively compact open set A in X,*

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\lambda_1 t} Z_t(A) < \infty\right) = 1, \quad x \in X. \tag{3.21}$$

As a consequence, for any $\kappa < \lambda_1$,

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\kappa t} Z_t(A) = 0\right) = 1, \quad x \in X.$$

(ii) *Assume that the support of the branching rate μ is compact. Then, for any non-empty open set A in X such that $P_x(\rho_A < \infty) = 1$ for any $x \in X$ and $\kappa > \lambda_1$,*

$$\mathbf{P}_x \left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty \,\Big|\, L_A = \infty \right) = 1, \quad x \in X. \tag{3.22}$$

Proof. Let A be a relatively compact open set in X. Then

$$e^{\lambda_1 t} Z_t(A) \leq \frac{1}{\inf_{x\in A} h(x)} M_t.$$

Since

$$\limsup_{t\to\infty} e^{\lambda_1 t} Z_t(A) \leq \frac{1}{\inf_{x\in A} h(x)} M_\infty < \infty \quad \mathbf{P}_x\text{-a.s.},$$

(3.21) holds. The equation (3.22) follows from Proposition 3.7. □

Remark 3.9. Engländer and Kyprianou [26] studied the exponential growth of the numbers of particles in every relatively compact open set for branching diffusion processes such that the branching rates are nonnegative, bounded and continuous functions. On the other hand, we can take unbounded functions as branching rates in (3.20) of Proposition 3.7 and Theorem 3.8 (i). For instance, let us consider a branching Brownian motion on $\mathbb{R}^3$. Then, since the measure $\mu(dx) = 1/|x|\chi_{|x|\leq 1}\,dx$ belongs to $\mathcal{K}_\infty^{\mathbb{R}^3}$, we can take the measure μ as branching rate. Moreover, the ground state of $\lambda_1(\mu;\mathbb{R}^3)$ satisfies (1.24) because the support of μ is compact.

Assume that $\mathbf{M}$ is Harris recurrent. Let us consider the branching symmetric Hunt process $\overline{\mathbf{M}} = (\mathbf{P}_x)$ on X such that the branching rate μ belongs to $\mathcal{K}_\infty$. Denote by T the first splitting time of $\overline{\mathbf{M}}$. Since $P_x(A_\infty^\mu = \infty) = 1$ for any $x \in X$ (see [46, p.426, Proposition 3.11]), it follows that

$$\mathbf{P}_x(T = \infty) = E_x\left[\exp\left(-A_\infty^\mu\right)\right] = 0$$

for any $x \in X$. Using this fact, we can show Theorem 3.4 (i), (ii) and Theorem 3.8 by the same argument. Here the condition $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy) < \infty$ is replaced with $\mu(X) < \infty$ and the condition on the lifetime or the last exit times is not imposed.

Remark 3.10. Let $\mathbf{M}^D$ be an absorbing symmetric α-stable process on an open set D in $\mathbb{R}^d$ and assume that $\mathbf{M}^D$ is transient. As we mentioned in Remark 2.12, any measure $\mu \in \mathcal{S}_\infty^D$ satisfies $\mu|_O \in \mathcal{S}_\infty^O$ for every bounded $C^{1,1}$ domain O in D. Hence, if we take a branching rate $\mu \in \mathcal{S}_\infty^D$ such that $\iint_{D\times D} G^{\mu,D}(x,y)\,\mu(dx)\mu(dy) < \infty$, then the arguments from Lemma 3.5 to Theorem 3.8 work, where $G^{\mu,D}(x,y)$ is the Green function of the $\exp(-A_t^\mu)$-subprocess of $\mathbf{M}^D$, that is,

$$\int_D G^{\mu,D}(x,y) f(y)\,dy = E_x\left[\int_0^{\tau_D} \exp\left(-A_t^\mu\right) f(X_t)\,dt\right].$$

3.2 Examples

We apply Theorems 3.4 and 3.8 to branching Brownian motions and branching symmetric α-stable processes. Let $\mathbf{M}^D$ be an absorbing symmetric α-stable process on an open set D and $(\mathcal{E}^D, \mathcal{F}^D)$ the associated Dirichlet form. Recall that

$$\lambda_1(\mu; D) = \inf\left\{\mathcal{E}^D(u,u) - \int_D u^2\,d\mu : u \in C_0^\infty(D), \int_D u^2\,dx = 1\right\}$$

for $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty^D - \mathcal{K}_\infty^D$. Let us denote by $\overline{\mathbf{M}^D} = (\mathbf{X}_t, \mathbf{P}_x)$ the branching symmetric α-stable process such that the motion component is $\mathbf{M}^D$ and the branching rate μ belongs to the class $\mathcal{K}_\infty^D$.

Example 3.11. Let $\overline{\mathbf{M}^\alpha} = (\mathbf{X}_t, \mathbf{P}_x)$ be a branching symmetric α-stable process on $\mathbb{R}$ with branching rate δ_0. Assume that the branching mechanism satisfies $p_0(0) + p_2(0) = 1$. Then $Q(0) = 2p_2(0)$. Since the extinction probability is a minimal solution to (2.1) as can be proved in a way similar to that yielding Proposition 2.1, we obtain

$$\mathbf{P}_x(e_0 < \infty) = \begin{cases} 1, & 0 \leq p_2(0) \leq 1/2 \\ (1 - p_2(0))/p_2(0), & 1/2 < p_2(0) \leq 1. \end{cases}$$

Hence if $1/2 < p_2(0) \leq 1$, then it holds that

$$\mathbf{P}_x\left(\lim_{t\to\infty} e^{\lambda_1(\alpha)t} \sum_{i=1}^{Z_t} h(\mathbf{X}_t^i) \in (0, \infty) \,\middle|\, e_0 = \infty \right) = 1$$

and

$$\mathbf{P}_x\left(\liminf_{t\to\infty} e^{\lambda_1(\alpha)t} Z_t > 0 \,\middle|\, e_0 = \infty \right) = 1,$$

where the principal eigenvalue $\lambda_1(\alpha) := \lambda_1((Q-1)\delta_0; \mathbb{R})$ and the corresponding ground state h are the same as in Example 6.10 below with Q there replaced by $Q(0)$, respectively. It also holds that, for any relatively compact open set A,

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\lambda_1(\alpha)t} Z_t(A) < \infty \right) = 1$$

and

$$\mathbf{P}_x\left(\limsup_{t\to\infty} e^{\kappa t} Z_t(A) = \infty \,\middle|\, L_A = \infty \right) = 1$$

for any $\kappa > \lambda_1(\alpha)$.

Example 3.12. Suppose that $d = 1$ and $1 < \alpha \leq 2$. Let us take first $D = (-R, R)$ and $\mu = \delta_a,\ a \in (-R, R)$. We then see from Example 6.5 below and (1.17) that

$$\lambda_1(\delta_a; (-R, R)) < 0 \iff R > \frac{A + \sqrt{A^2 + 4a^2}}{2}, \tag{3.23}$$

where

$$A = \left\{ (\alpha - 1) 2^{\alpha-2} \Gamma\left(\frac{\alpha}{2}\right)^2 \right\}^{1/(\alpha-1)}.$$

Note that $\lim_{R\to\infty} \lambda_1(\delta_a; (-R, R)) = \lambda_1(\delta_0; \mathbb{R})$ for each $a \in \mathbb{R}$. Let $\lambda_1 = \lambda_1(\delta_a; (-R, R))$ and denote by h the corresponding ground state. Then

$$h(x) = G^R_{-\lambda_1}(x, a) h(a),$$

where $G^R_{-\lambda_1}(x, y)$ is the $-\lambda_1$-resolvent of the absorbing symmetric α-stable process on $(-R, R)$. It follows from (3) of [45] and (1.24) that, if $1 < \alpha < 2$, then

$$h(x) = \begin{cases} O\left((R - x)^{\alpha/2}\right), & x \to R \\ O\left((R + x)^{\alpha/2}\right), & x \to -R. \end{cases} \tag{3.24}$$

Let us consider the binary branching absorbing symmetric α-stable process on $(-R,R)$ with branching rate δ_a. Then this process does not extinct if and only if a and R satisfies the right hand side of (3.23) by Theorem 2.4. Note that $(-R,R)$ is Green bounded because

$$E_x[\tau_R] = \frac{2}{\Gamma(\alpha+1)}(R^2-x^2)^{\alpha/2}$$

as proved by Getoor [30, Section 5] or S. Watanabe [60, Theorem 2.1], where τ_R is the exit time of the one-dimensional symmetric α-stable process from $(-R,R)$. Therefore, if a and R satisfies (3.23), then (3.7), (3.9) and (3.12) hold for this process.

Next consider the binary branching absorbing symmetric α-stable process on $(0,\infty)$ with branching rate δ_a. Then

$$\lambda_1(\delta_a;(0,\infty))<0 \iff a > \left\{\frac{(\alpha-1)\Gamma\left(\frac{\alpha}{2}\right)^2}{2}\right\}^{1/(\alpha-1)} \tag{3.25}$$

by Example 6.5 below and (1.17). This condition is also equivalent to say that the branching process does not extinct by Theorem 2.4. Denote by h the ground state corresponding to $\lambda_1(\delta_a;(0,\infty))$. Then it follows from (3) of [45] and (1.24) that

$$h(x) = \begin{cases} O\left(x^{\alpha/2}\right), & x\to 0 \\ O\left(x^{-(1+\alpha)}\right), & x\to\infty. \end{cases}$$

Since $(0,\infty)$ is not Green bounded, (3.7), (3.9) and (3.11) hold if a satisfies (3.25).

Example 3.13. Suppose that $1<\alpha\le 2$ and $d>\alpha$. Let us take $D=\mathbb{R}^d$ and $\mu=\delta_R$, the surface measure on $\{x\in\mathbb{R}^d : |x|=R\}$. Then it follows from Example 4.1 of [58] and (1.17) that

$$\lambda_1(\delta_R;\mathbb{R}^d)<0 \iff R > \left\{\frac{\sqrt{\pi}\,\Gamma\left(\frac{d+\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)}\right\}^{1/(\alpha-1)}. \tag{3.26}$$

Hence, the binary branching symmetric α-stable process on $\mathbb{R}^d$ with branching rate δ_R does not extinct locally if and only if R satisfies the right hand side of (3.26). Under this condition, (3.21) and (3.22) hold for this process.

Chapter 4

Limit theorems for branching symmetric Markov processes

We establish limit theorems for branching symmetric Hunt processes by using the principal eigenvalues and the ground states of associated Schrödinger operators. We apply them to branching Brownian motions and branching symmetric α-stable processes.

4.1 h-transform and ergodicity

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbf{M} = (X_t, P_x)$ be an m-symmetric Hunt process on X. In this section, we fix a signed measure μ which can be decomposed into $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty - \mathcal{K}_\infty$. Assume that $\mathbf{M}$ satisfies Assumptions 1.3 and 1.5 and that $\lambda_1 := \lambda_1(\mu) < 0$. Let h be the normalized positive eigenfunction corresponding to λ_1 with $\int_X h^2 \, dm = 1$.

First recall the notations from Section 1.1: M_t^u is the martingale additive functional and N_t^u is the continuous additive functional of zero energy for $u \in \mathcal{F}_e$ as appeared in (1.7). $M_t^{u,c}$ is the continuous part of M_t^u and $\mu_{\langle M^{u,c}\rangle}$ is the energy measure of $M_t^{u,c}$. The measure $J(dx, dy)$ is the jump measure of $\mathbf{M}$ defined in (1.6). We now suppose that $\mu^- = 0$. Since it holds that $h = G_{-\lambda_1}(h\mu)$ on X, Fukushima's decomposition (1.7) implies that for q.e. $x \in X$, P_x-a.s.

$$\begin{aligned} h(X_t) - h(X_0) &= M_t^h + N_t^h \\ &= M_t^h - \lambda_1 \int_0^t h(X_s) \, ds - \int_0^t h(X_s) \, dA_s^\mu, \quad t \geq 0. \end{aligned}$$

Set

$$M_t = \int_0^t \frac{1}{h(X_{s-})} \, dM_s^h.$$

Then the solution R_t to the stochastic differential equation

$$R_t = 1 + \int_0^t R_{s-} \, dM_s$$

is a positive local martingale, and thus a supermartingale. As a result, the formula

$$dP_x^h = R_t \, dP_x \quad \text{on } \mathcal{F}_t$$

uniquely determines a family of probability measures $\left\{P_x^h, x \in X\right\}$ on $(\Omega, \mathcal{F}_\infty)$. To emphasize, the Hunt process $\mathbf{M}$ under probability measures $\left\{P_x^h, x \in X\right\}$ will be denoted by $\mathbf{M}^h$; that is,

$$E_x^h\left[f(X_t^h)\right] := E_x\left[R_t f(X_t)\right]$$

for $t > 0$ and $f \in \mathcal{B}^+(X)$. It follows from [16] that the process $\mathbf{M}^h$ is h^2m-symmetric and irreducible because $\exp\left(A_t^\mu\right) h(X_t)$ is strictly positive. If $\mu^- \neq 0$, then we can also apply the same argument as above to the $\exp\left(-A_t^{\mu^-}\right)$-subprocess of $\mathbf{M}$. Let $(\mathcal{E}^h, \mathcal{F}^h)$ be the symmetric Dirichlet form on $L^2(X; h^2m)$ associated with $\mathbf{M}^h$. We then have by Theorems 2.6 and 2.8 of [16] the following:

Theorem 4.1. (i) *It holds that*

$$\begin{aligned}
\mathcal{E}^h(u,u) &= \frac{1}{2}\int_X h(x)^2\,\mu_{\langle M^{u,c}\rangle}(dx) + \iint_{X\times X\setminus\triangle}(u(x)-u(y))^2\,h(x)h(y)J(dx,dy),\\
\mathcal{F}^h &= \overline{\mathcal{F}}^{\mathcal{E}_1^h(\cdot,\cdot)},
\end{aligned}$$

where $\mathcal{E}_1^h(u,u) = \mathcal{E}^h(u,u) + \int_X u^2\,h^2dm$.
(ii) *The constant function* $\mathbf{1}$ *belongs to* $\mathcal{F}^h$ *and* $\mathcal{E}^h(\mathbf{1},\mathbf{1}) = 0$. *Consequently,* $\mathbf{M}^h$ *is recurrent.*

Note that, by Doléan-Dade's formula (see [33, Theorem 9.39]),

$$R_t = \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right)\prod_{0<s\le t}\frac{h(X_s)}{h(X_{s-})}\exp\left(1-\frac{h(X_s)}{h(X_{s-})}\right),$$

where M_t^c is a continuous part of M_t. Then, applying Ito's formula [33, Theorem 9.35] to $\log h(X_t)$, we obtain for q.e. $x \in X$, P_x-a.s.

$$\log h(X_t) - \log h(X_0) = M_t - \frac{1}{2}\langle M^c\rangle_t + \sum_{s\le t}\left(\log\frac{h(X_s)}{h(X_{s-})} + \frac{h(X_s)-h(X_{s-})}{h(X_{s-})}\right) - \lambda_1 t - A_t^\mu.$$

Hence

$$R_t = \exp\left(\lambda_1 t + A_t^\mu\right)\frac{h(X_t)}{h(X_0)},$$

that is,

$$E_x^h\left[f(X_t^h)\right] = \frac{e^{\lambda_1 t}}{h(x)}E_x\left[\exp\left(A_t^\mu\right)h(X_t)f(X_t)\right]. \tag{4.1}$$

(4.1) implies that $u \in \mathcal{F}^h$ if and only if $uh \in \mathcal{F}$ and that

$$\begin{aligned}
\mathcal{E}^h(u,u) &= \mathcal{E}(uh,uh) - \int_X (uh)(x)^2(\lambda_1 m(dx) + \mu(dx))\\
&= \mathcal{E}^\mu(uh,uh) - \lambda_1\int_X (uh)(x)^2 m(dx).
\end{aligned}$$

In other words, $\Phi^h : u \mapsto uh$ is an isometry from $(\mathcal{E}^h, \mathcal{F}^h)$ onto $(\mathcal{E}^{\mu+\lambda_1 m}, \mathcal{F})$ and from $L^2(X; h^2m)$ onto $L^2(X; m)$.

By Theorem 4.1 (ii),

$$\inf\left\{\mathcal{E}^h(u,u) : u \in \mathcal{F}^h,\ \int_X u^2\, h^2\, dm = 1\right\} = 0.$$

Let $\lambda_2^h := \lambda_2^h(\mu)$ be the spectral gap for the self-adjoint operator associated with $(\mathcal{E}^h, \mathcal{F}^h)$,

$$\lambda_2^h(\mu) := \inf\left\{\mathcal{E}^h(u,u) : u \in \mathcal{F}^h,\ \int_X u^2\, h^2 dm = 1,\ \int_X u\, h^2 dm = 0\right\}.$$

Since all the spectra are invariant under the isometry Φ^h, it follows that

$$\lambda_2^h = \lambda_2(\mu) - \lambda_1(\mu) > 0, \tag{4.2}$$

which leads us to the following Poincaré inequality:

$$\|p_t^h \varphi\|_{L^2(X;h^2m)} \le e^{-\lambda_2^h t}\|\varphi\|_{L^2(X;h^2m)} \tag{4.3}$$

for any $\varphi \in L^2(X; h^2m)$ with $\int_X \varphi\, h^2 dm = 0$. Here $\{p_t^h, t \ge 0\}$ is the Markovian transition semigroup of $\mathbf{M}^h$,

$$p_t^h f(x) = E_x^h[f(X_t^h)], \quad f \in \mathcal{B}^+(X).$$

Note that p_t^h has the transition density kernel $p_t^h(x,y)$ with respect to the measure h^2m given by

$$p_t^h(x,y) = e^{\lambda_1 t}\frac{p_t^\mu(x,y)}{h(x)h(y)}.$$

Then

$$\begin{aligned}
\left|p_{s+t}^h \varphi(x)\right| &= \left|p_s^h\left(p_t^h\varphi\right)(x)\right| \\
&= \left|\frac{e^{\lambda_1 s}}{h(x)}\int_X p_s^\mu(x,y)p_t^h\varphi(y)h(y)\, m(dy)\right| \\
&= \frac{e^{\lambda_1 s}}{h(x)}\frac{\left|p_s^\mu\left(\left(p_t^h\varphi\right)\cdot h\right)(x)\right|}{\|(p_t^h\varphi)\cdot h\|_{L^2(X;m)}}\|p_t^h\varphi\|_{L^2(X;h^2m)} \\
&\le \frac{e^{\lambda_1 s}}{h(x)}\|p_s^\mu\|_{2,\infty}\|p_t^h\varphi\|_{L^2(X;h^2m)}.
\end{aligned}$$

Hence, for every $\varphi \in L^2(X; h^2m)$ with $\int_X \varphi\, h^2 dm = 0$, we see from (4.3) that the last term above is not greater than

$$\frac{e^{\lambda_1 s}}{h(x)}\|p_s^\mu\|_{2,\infty}e^{-\lambda_2^h t}\|\varphi\|_{L^2(X;h^2m)}.$$

Taking $s = 1/2$, we see that there exists a constant $C > 0$ such that

$$|h(x)p_t^h\varphi(x)| \le Ce^{-\lambda_2^h t}\|\varphi\|_{L^2(X;h^2m)} \qquad \text{for } t \ge 1 \tag{4.4}$$

for every $\varphi \in L^2(X; h^2m)$ with $\int_X \varphi\, h^2 dm = 0$. For $\varphi \in L^2(X; h^2m)$, we can write $\varphi(x) = \int_X \varphi\, h^2 dm + \varphi_0(x)$, where $\varphi_0(x) = \varphi(x) - \int_X \varphi\, h^2 dm$ has the property that $\varphi_0 \in L^2(X; h^2m)$ with $\int_X \varphi_0\, h^2 dm = 0$. As $p_t^h 1 = 1$, we see from (4.4) that

$$\lim_{t\to\infty} p_t^h\varphi(x) = \int_X \varphi\, h^2 dm + \lim_{t\to\infty} p_t^h\varphi_0(x) = \int_X \varphi\, h^2 dm, \qquad x \in X \tag{4.5}$$

for any $\varphi \in L^2(X; h^2 m)$. This together with (4.1) yields that

$$\lim_{t\to\infty} e^{\lambda_1 t} E_x \left[\exp\left(A_t^\mu\right) f(X_t)\right] = h(x) \int_X fh\, dm, \qquad x \in X \tag{4.6}$$

for any $f \in L^2(X; m)$.

4.2 Limit theorems

Throughout this section, we assume that $\mathbf{M}$ satisfies Assumptions 1.3 and 1.5. Let $\overline{\mathbf{M}} = (\mathbf{\Omega}, \mathcal{G}, \mathcal{G}_t, \mathbf{X}_t, \mathbf{P}_x)$ be the branching symmetric Hunt process with motion component $\mathbf{M}$, branching rate $\mu \in \mathcal{K}_\infty$ and branching mechanism $\{p_n(x)\}_{n\geq 0}$. We assume that $\sup_{x\in X} Q(x) < \infty$ and $\lambda_1 := \lambda_1((Q-1)\mu) < 0$, where $Q(x) = \sum_{n=0}^\infty n p_n(x)$.

Lemma 4.2. *It holds that*

$$\lim_{t\to\infty} e^{\lambda_1 t} \mathbf{E}_x \left[Z_t(f)\right] = h(x) \int_X fh\, dm, \quad x \in X \tag{4.7}$$

for any $f \in L^2(X; m)$.

Proof. Since it follows from (1.19) that

$$e^{\lambda_1 t} \mathbf{E}_x \left[Z_t(f)\right] = e^{\lambda_1 t} E_x \left[\exp\left(A_t^{(Q-1)\mu}\right) f(X_t)\right], \tag{4.8}$$

we get (4.7) from (4.6). □

Recall that

$$M_t = e^{\lambda_1 t} Z_t(h), \qquad t \geq 0 \tag{4.9}$$

and there exists a limit $M_\infty = \lim_{t\to\infty} M_t \in [0, \infty)$ $\mathbf{P}_x$-a.s. In the sequel, we assume that $\sup_{x\in X} R(x) < \infty$. We then obtain

Proposition 4.3. (i) *It holds that*

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(f) = M_\infty \int_X fh\, dm \qquad \text{in } \mathbf{P}_x\text{-probability} \tag{4.10}$$

for any $f \in L^2(X; m) \cap \mathcal{B}_b(X)$.
(ii) *Let $\{t_n\}$ be any sequence such that $\sum_{n=1}^\infty e^{-\varepsilon t_n} < \infty$ for some positive $\varepsilon > 0$ so that $0 < \varepsilon < (-\lambda_1) \wedge 2\lambda_2^h$. Then*

$$\lim_{n\to\infty} e^{\lambda_1 t_n} Z_{t_n}(f) = M_\infty \int_X fh\, dm \qquad \mathbf{P}_x\text{-a.s.} \tag{4.11}$$

for any $f \in L^2(X; m) \cap \mathcal{B}_b(X)$.

Proof. Let $f \in L^2(X; m) \cap \mathcal{B}_b(X)$ and $g(x) = f(x) - h(x) \int_X fh\, dm$. Then

$$e^{\lambda_1 t} Z_t(f) = M_t \int_X fh\, dm + e^{\lambda_1 t} Z_t(g).$$

By (1.20),

$$\mathbf{E}_x\left[\left(e^{\lambda_1 t}Z_t(g)\right)^2\right] = \mathrm{I}+\mathrm{II}, \tag{4.12}$$

where

$$\mathrm{I} := e^{2\lambda_1 t}E_x\left[\exp\left(A_t^{(Q-1)\mu}\right)g(X_t)^2\right]$$

and

$$\mathrm{II} := E_x\left[\int_0^t \exp\left(2\lambda_1 s + A_s^{(Q-1)\mu}\right)\left(e^{\lambda_1(t-s)}E_{X_s}\left[\exp\left(A_{t-s}^{(Q-1)\mu}\right)g(X_{t-s})\right]\right)^2 dA_s^{R\mu}\right].$$

Recall that $\lambda_2^h := \lambda_2^h((Q-1)\mu) > 0$ as we saw in (4.2). Then, since for any positive $\varepsilon < (-\lambda_1)\wedge(2\lambda_2^h)$,

$$\sup_{(x,t)\in X\times(0,\infty)} E_x\left[\exp\left((2\lambda_1+\varepsilon)t + A_t^{(Q-1)\mu}\right)\right] < \infty$$

by Theorem 5.1 of [14] or Theorem 2.4 of [52], it follows that

$$\mathrm{I} \le e^{-\varepsilon t}E_x\left[\exp\left((2\lambda_1+\varepsilon)t + A_t^{(Q-1)\mu}\right)\right]\|g\|^2_{L^\infty(X;m)} \le c_1\, e^{-\varepsilon t}\,\|g\|^2_{L^\infty(X;m)}.$$

By (4.4),

$$\begin{aligned}
\mathrm{II} &\le c_2 E_x\left[\int_0^t \exp\left(2\lambda_1 s + A_s^{(Q-1)\mu}\right)e^{-2\lambda_2^h(t-s)}\,dA_s^{R\mu}\right]\|g\|^2_{L^2(X;m)} \\
&\le c_2 e^{-\varepsilon t}E_x\left[\int_0^\zeta \exp\left((2\lambda_1+\varepsilon)s + A_s^{(Q-1)\mu}\right)dA_s^{R\mu}\right]\|g\|^2_{L^2(X;m)}.
\end{aligned}$$

Since

$$\inf\left\{\mathcal{E}(u,u) - \int_X u^2(Q-1)\,d\mu - (2\lambda_1+\varepsilon)\int_X u^2\,dm : u\in\mathcal{F}, \int_X u^2\,dm = 1\right\} = -\lambda_1-\varepsilon > 0$$

by the definition of λ_1, we deduce from Theorem 1.2 and [52, Lemma 3.5] that

$$\sup_{x\in X} E_x\left[\int_0^\zeta \exp\left((2\lambda_1+\varepsilon)s + A_s^{(Q-1)\mu}\right)dA_s^{R\mu}\right] < \infty,$$

and thus $\mathrm{II} \le c_3\, e^{-\varepsilon t}\,\|g\|^2_{L^\infty(X;m)}$. Combining these estimates implies that

$$\mathbf{E}_x\left[\left(e^{\lambda_1 t}Z_t(g)\right)^2\right] \le (c_1+c_3)\,e^{-\varepsilon t}\,\|g\|^2_{L^\infty(X;m)}. \tag{4.13}$$

Furthermore, by Chebyshev's inequality,

$$\begin{aligned}
\mathbf{P}_x\left(\left|e^{\lambda_1 t}Z_t(g)\right| > \delta\right) &\le \frac{1}{\delta^2}\mathbf{E}_x\left[\left(e^{\lambda_1 t}Z_t(g)\right)^2\right] \\
&\le \frac{C}{\delta^2}e^{-\varepsilon t}\,\|g\|^2_{L^\infty(X;m)}
\end{aligned}$$

for any $\delta > 0$, and the last term above goes to 0 as $t\to\infty$. Therefore (4.10) follows.

Let f and g be the same as above, respectively. By (4.13),

$$\sum_{n=1}^{\infty} \mathbf{E}_x \left[\left(e^{\lambda_1 t_n} Z_t(g) \right)^2 \right] \le C \sum_{n=1}^{\infty} e^{-\varepsilon t_n} < \infty.$$

By Borel-Cantelli's lemma, we have $\lim_{n\to\infty} e^{\lambda_1 t_n} Z_{t_n}(g) = 0$ $\mathbf{P}_x$-a.s., and so (4.11) as $\lim_{t\to\infty} M_t = M_\infty$ $\mathbf{P}_x$-a.s. $\square$

We will assume the following on X and the branching rate:

Assumption 4.4. Either (i) or (ii) holds:
(i) It holds that $P_x(\zeta < \infty) = 1$ for every $x \in X$ and the branching rate $\mu \in \mathcal{K}_\infty$ satisfies $\iint_{X\times X} G^\mu(x,y)\,\mu(dx)\mu(dy) < \infty$.
(ii) $\mathbf{M}$ is Harris recurrent and the branching rate $\mu \in \mathcal{K}_\infty$ satisfies $\mu(X) < \infty$.

Recall that we proved Lemma 3.1 and Proposition 3.3 that M_t in (4.9) is a square integrable martingale and that

$$\{e_0 = \infty\} = \{M_\infty \in (0,\infty)\} \quad \mathbf{P}_x\text{-a.s.}, \tag{4.14}$$

where e_0 is the *extinction time* of $\overline{\mathbf{M}}$ defined by

$$e_0 = \inf\{t > 0 : Z_t = 0\}.$$

We then get the following immediately from the above, Lemma 4.2 and Proposition 4.3.

Corollary 4.5. (i) *It holds that*

$$\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x[Z_t(A)]} = \frac{M_\infty}{h(x)} \qquad \textit{in } \mathbf{P}_x\textit{-probability}$$

for every Borel subset A in X such that $0 < m(A) < \infty$.
(ii) *Suppose that Assumption* 4.4 *holds. If $m(X) < \infty$, then*

$$\lim_{t\to\infty} \frac{Z_t(A)}{Z_t} = \frac{\int_A h\,dm}{\int_X h\,dm} \qquad \textit{in } \mathbf{P}_x^e\textit{-probability}$$

for every Borel subset A in X, where $\mathbf{P}_x^e(\cdot) = \mathbf{P}_x(\cdot \mid e_0 = \infty)$.
(iii) *Suppose that Assumption* 4.4 *holds. Let $\{t_n\}$ be any sequence as in Proposition* 4.3. *If $m(X) < \infty$, then*

$$\lim_{n\to\infty} \frac{Z_{t_n}(A)}{Z_{t_n}} = \frac{\int_A h\,dm}{\int_X h\,dm} \qquad \mathbf{P}_x^e\textit{-a.s.}$$

for every Borel subset A in X.

Proof. Let A be a Borel subset in X such that $0 < m(A) < \infty$. Then combining Proposition 4.3 with Lemma 4.2 implies that

$$\begin{aligned}\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x[Z_t(A)]} &= \lim_{t\to\infty} \frac{e^{\lambda_1 t} Z_t(A)}{e^{\lambda_1 t}\mathbf{E}_x[Z_t(A)]} \\ &= \frac{M_\infty \int_A h\,dm}{h(x)\int_A h\,dm} = \frac{M_\infty}{h(x)} \quad \text{in } \mathbf{P}_x\text{-probability},\end{aligned}$$

whence (i) holds. We now that assume that Assumption 4.4 holds and that $0 < m(X) < \infty$. Then, since the constant function belongs to $L^2(X; m)$, we obtain by Proposition 4.3 and (4.14),

$$\begin{aligned}\lim_{t\to\infty} \frac{Z_t(A)}{Z_t} &= \lim_{t\to\infty} \frac{e^{\lambda_1 t} Z_t(A)}{e^{\lambda_1 t} Z_t} \\ &= \frac{M_\infty \int_A h\,dm}{M_\infty \int_X h\,dm} = \frac{\int_A h\,dm}{\int_X h\,dm} \quad \text{in } \mathbf{P}^e_x\text{-probability,}\end{aligned}$$

which yields (ii). By the same way, (iii) follows. □

Corollary 4.5 is an extension of the result for branching Brownian motions by S. Watanabe [61, Corollary on p.397] to branching symmetric Hunt processes with state dependent branching rates and branching mechanisms.

Remark 4.6. Let $\mathbf{M}^\alpha = (X_t, P_x)$ be the symmetric α-stable process on $\mathbb{R}^d$. Since (4.6) is true for any $f \in \mathcal{B}_b(\mathbb{R}^d)$ by [55, Corollary 4.7], Lemma 4.2 holds for any $f \in \mathcal{B}_b(\mathbb{R}^d)$. We now consider the branching symmetric α-stable process with motion component $\mathbf{M}^\alpha$ and branching rate $\mu \in \mathcal{K}_\infty^{\mathbb{R}^d}$. Then for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\begin{aligned}&\sup_{(x,t)\in\mathbb{R}^d\times(1,\infty)} \left| e^{\lambda_1 t} E_x \left[\exp\left(A_t^{(Q-1)\mu} \right) f(X_t) \right] \right| \\ &\le C_p \|f\|_{L^\infty(\mathbb{R}^d;dx)} \sup_{x\in\mathbb{R}^d} \|h(x) p_1^h(x,\cdot)\|_{L^p(\mathbb{R}^d;h^2dx)} \left\| \frac{1}{h} \right\|_{L^q(\mathbb{R}^d;h^2dx)} < \infty\end{aligned} \tag{4.15}$$

for any $p > 2 + n/\alpha$ and $q = p/(p-1)$ by Lemmas 4.4 and 4.6 of [55], where C_p is a positive constant depending on p. Thus II in the proof of Proposition 4.3 converges to 0 as $t \to \infty$ for any $f \in \mathcal{B}_b(\mathbb{R}^d)$ by combining (4.15) with the dominated convergence theorem, instead of the inequality (4.4). As a result, (4.10) holds for any $f \in \mathcal{B}_b(\mathbb{R}^d)$, which leads us to that Corollary 4.5 (i) and (ii) hold for every Borel subset A in $\mathbb{R}^d$.

We are now in a position to establish the following almost sure convergence of $e^{\lambda_1 t} Z_t(f)$.

Theorem 4.7. *There exists a subspace $\mathbf{\Omega}_0 \subset \mathbf{\Omega}$ of full probability such that, for every $\omega \in \mathbf{\Omega}_0$ and for every bounded Borel measurable function f on X with compact support whose set of discontinuous points has zero m-measure,*

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(f)(\omega) = M_\infty(\omega) \int_X f h\,dm. \tag{4.16}$$

Observe that h is strictly positive and continuous on X. So every bounded Borel measurable function f with compact support is bounded by ch for some $c > 0$.

Our approach to Theorem 4.7 is similar to that to [4, Theorem 1']. We now prove two lemmas. Let δ be a positive constant $0 < \delta < (-\lambda_1) \wedge 2\lambda_2^h$ and denote by $\mathbf{X}_t^{n\delta,i}$ the particles at time $t \ge n\delta$ such that whose parent is $\mathbf{X}^i_{n\delta}$. Let U be a nearly Borel subset of X, and for $x \in X$ and $\varepsilon > 0$,

$$U^\varepsilon(x) := \left\{ y \in U : \, h(y) \ge \frac{1}{1+\varepsilon} h(x) \right\}.$$

Define

$$Y_{n,i}^{\delta,\varepsilon} = \frac{1}{1+\varepsilon} h(\mathbf{X}^i_{n\delta}) \mathbf{1}_{\left\{\mathbf{X}_t^{n\delta,i} \in U^\varepsilon(\mathbf{X}^i_{n\delta}) \text{ for every } t\in[n\delta,(n+1)\delta]\right\}}$$

and $S_n^{\delta,\varepsilon} = e^{\lambda_1 n\delta} \sum_{i=1}^{Z_{n\delta}} Y_{n,i}^{\delta,\varepsilon}$.

Lemma 4.8. *It holds that*

$$\lim_{n\to\infty}\left(S_n^{\delta,\varepsilon}-\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\right)=0\qquad \mathbf{P}_x\text{-}a.s.$$

Proof. A direct calculation implies that

$$\begin{aligned}\mathbf{E}_x\left[\left(S_n^{\delta,\varepsilon}-\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\right)^2\right]&=\mathbf{E}_x\left[\left(S_n^{\delta,\varepsilon}\right)^2-2S_n^{\delta,\varepsilon}\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]+\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]^2\right]\\&=\mathbf{E}_x\left[\mathbf{E}_x\left[\left(S_n^{\delta,\varepsilon}\right)^2\,\middle|\,\mathcal{G}_{n\delta}\right]-\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]^2\right].\end{aligned}\tag{4.17}$$

Since

$$\begin{aligned}\mathbf{E}_x\left[\left(S_n^{\delta,\varepsilon}\right)^2\,\middle|\,\mathcal{G}_{n\delta}\right]&=e^{2\lambda_1 n\delta}\mathbf{E}_x\left[\sum_{i=1}^{Z_{n\delta}}\left(Y_{n,i}^{\delta,\varepsilon}\right)^2+\sum_{1\le i,j\le Z_{n\delta},i\neq j}Y_{n,i}^{\delta,\varepsilon}Y_{n,j}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\\&=e^{2\lambda_1 n\delta}\sum_{i=1}^{Z_{n\delta}}\mathbf{E}_x\left[\left(Y_{n,i}^{\delta,\varepsilon}\right)^2\,\middle|\,\mathcal{G}_{n\delta}\right]+e^{2\lambda_1 n\delta}\sum_{1\le i,j\le Z_{n\delta},i\neq j}\mathbf{E}_x\left[Y_{n,i}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\mathbf{E}_x\left[Y_{n,j}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]^2&=\left(e^{\lambda_1 n\delta}\sum_{i=1}^{Z_{n\delta}}\mathbf{E}_x\left[Y_{n,i}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\right)^2\\&=e^{2\lambda_1 n\delta}\sum_{i=1}^{Z_{n\delta}}\mathbf{E}_x\left[Y_{n,i}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]^2+e^{2\lambda_1 n\delta}\sum_{1\le i,j\le Z_{n\delta},i\neq j}\mathbf{E}_x\left[Y_{n,i}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\mathbf{E}_x\left[Y_{n,j}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right],\end{aligned}$$

the last term of (4.17) is equal to

$$e^{2\lambda_1 n\delta}\mathbf{E}_x\left[\sum_{i=1}^{Z_{n\delta}}\left(\mathbf{E}_x\left[\left(Y_{n,i}^{\delta,\varepsilon}\right)^2\,\middle|\,\mathcal{G}_{n\delta}\right]-\mathbf{E}_x\left[Y_{n,i}^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]^2\right)\right]\le e^{2\lambda_1 n\delta}\mathbf{E}_x\left[\sum_{i=1}^{Z_{n\delta}}\mathbf{E}_x\left[\left(Y_{n,i}^{\delta,\varepsilon}\right)^2\,\middle|\,\mathcal{G}_{n\delta}\right]\right].$$

By the Markov property and (1.19), the last term above is equal to

$$\begin{aligned}e^{2\lambda_1 n\delta}\mathbf{E}_x\left[\sum_{i=1}^{Z_{n\delta}}\mathbf{E}_{\mathbf{X}_{n\delta}^i}\left[\left(Y_{0,1}^{\delta,\varepsilon}\right)^2\right]\right]&=e^{2\lambda_1 n\delta}E_x\left[\exp\left(A_{n\delta}^{(Q-1)\mu}\right)\mathbf{E}_{X_{n\delta}}\left[\left(Y_{0,1}^{\delta,\varepsilon}\right)^2\right]\right]\\&\le e^{2\lambda_1 n\delta}E_x\left[\exp\left(A_{n\delta}^{(Q-1)\mu}\right)h(X_{n\delta})^2\right]\\&\le e^{2\lambda_1 n\delta}E_x\left[\exp\left(A_{n\delta}^{(Q-1)\mu}\right)h(X_{n\delta})\right]\|h\|_{L^\infty(X;m)}\\&=e^{\lambda_1 n\delta}h(x)\|h\|_{L^\infty(X;m)}.\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}\mathbf{E}_x\left[\left(S_n^{\delta,\varepsilon}-\mathbf{E}_x\left[S_n^{\delta,\varepsilon}\,\middle|\,\mathcal{G}_{n\delta}\right]\right)^2\right]<\infty,$$

which yields the desired result by an application of Borel-Cantelli's lemma. □

Lemma 4.9. *It holds that*

$$\liminf_{t\to\infty} e^{\lambda_1 t} Z_t(\mathbf{1}_U h) \geq M_\infty \int_U h^2\, dm \qquad \mathbf{P}_x\text{-}a.s. \tag{4.18}$$

for every open subset U in X.

Proof. Since $e^{\lambda_1 t} Z_t(\mathbf{1}_U h) \geq e^{\lambda_1 \delta} S_n^{\delta,\varepsilon}$ for any $t \in [n\delta, (n+1)\delta]$, the Markov property and Lemma 4.8 yield that

$$\begin{aligned}
\liminf_{t\to\infty} e^{\lambda_1 t} Z_t(\mathbf{1}_U h) &\geq e^{\lambda_1\delta} \liminf_{n\to\infty} S_n^{\delta,\varepsilon} \\
&= e^{\lambda_1\delta} \liminf_{n\to\infty} e^{\lambda_1 n\delta} \sum_{i=1}^{Z_{n\delta}} \mathbf{E}_{\mathbf{X}^i_{n\delta}} \left[S_0^{\delta,\varepsilon} \right] \\
&= \frac{e^{\lambda_1\delta}}{1+\varepsilon} \liminf_{n\to\infty} e^{\lambda_1 n\delta} \sum_{i=1}^{Z_{n\delta}} h(\mathbf{X}^i_{n\delta}) \mathbf{P}_{\mathbf{X}^i_{n\delta}} \Big(\mathbf{X}_t \in U^\varepsilon(\mathbf{X}_0) \text{ for every } t \in [0,\delta] \Big).
\end{aligned}$$

By (4.11), the right hand side above is equal to

$$\begin{aligned}
&\frac{e^{\lambda_1\delta}}{1+\varepsilon} M_\infty \int_X \mathbf{P}_x\big(\mathbf{X}_t \in U^\varepsilon(\mathbf{X}_0), \text{ for every } t \in [0,\delta]\big) h(x)^2\, m(dx) \\
&= \frac{e^{\lambda_1\delta}}{1+\varepsilon} M_\infty \int_X E_x \left[e^{-A^\mu_\delta}; \delta < \tau_\varepsilon \right] h(x)^2\, m(dx) \\
&\geq \frac{e^{\lambda_1\delta}}{1+\varepsilon} M_\infty \int_U E_x \left[e^{-A^\mu_\delta}; \delta < \tau_\varepsilon \right] h(x)^2\, m(dx),
\end{aligned}$$

where $\tau_\varepsilon = \inf\{t > 0 : X_t \notin U^\varepsilon(X_0)\}$. Since X_t is right continuous, the last term above converges to $M_\infty \int_U h^2\, dm$ by letting first $\delta \to 0$ and then $\varepsilon \to 0$, whence (4.18) holds. □

Proof of Theorem 4.7. Since X is a locally compact separable metric space, there exists a countable base $\mathcal{U}$ of open set $\{U_k, k \geq 1\}$ that is closed under finite union. By Lemma 4.9, there exists $\mathbf{\Omega}_0 \subset \mathbf{\Omega}$ of full probability so that for every $\omega \in \mathbf{\Omega}_0$,

$$\liminf_{t\to\infty} e^{\lambda_1 t} Z_t(\mathbf{1}_{U_k} h)(\omega) \geq M_\infty(\omega) \int_{U_k} h^2\, dm \qquad \text{for every } U_k \in \mathcal{U}.$$

For any open set U, there exists a sequence of increasing open sets $\{U_{n_k}, k \geq 1\}$ in $\mathcal{U}$ so that $\cup_{k=1}^\infty U_{n_k} = U$. We have for every $\omega \in \mathbf{\Omega}_0$,

$$\begin{aligned}
\liminf_{t\to\infty} e^{\lambda_1 t} Z_t(\mathbf{1}_U h)(\omega) &\geq \liminf_{t\to\infty} e^{\lambda_1 t} Z_t(\mathbf{1}_{U_{n_k}} h)(\omega) \\
&\geq M_\infty(\omega) \int_{U_{n_k}} h^2\, dm \qquad \text{for every } k \geq 1.
\end{aligned}$$

Passing $k \to \infty$ yields that

$$\liminf_{t\to\infty} e^{\lambda_1 t} Z_t(\mathbf{1}_U h)(\omega) \geq M_\infty(\omega) \int_U h^2\, dm. \tag{4.19}$$

We now consider (4.16) on $\{M_\infty > 0\}$. For each fixed $\omega \in \mathbf{\Omega}_0 \cap \{M_\infty > 0\}$, define the probability measures μ_t and μ on X respectively, by

$$\mu_t(A)(\omega) = \frac{e^{\lambda_1 t} Z_t(\mathbf{1}_A h)(\omega)}{M_t(\omega)} \qquad \text{and} \qquad \mu(A) = \int_A h^2\, dm, \qquad A \in \mathcal{B}(X)$$

for every $t \geq 0$. Note that the measure μ_t is well-defined for every $t \geq 0$. The inequality (4.19) tells us that μ_t converges weakly to μ (for example, see [25, Theorem 9.1 on p.164]). Since h is strictly positive and continuous on X, for every function f on X with compact support on X whose discontinuity set has zero m-measure (equivalently zero μ-measure), $\phi := f/h$ is a bounded function having compact support with the same set of discontinuity with f. We thus have

$$\lim_{t\to\infty} \int_X \phi\, d\mu_t = \int_X \phi\, d\mu, \tag{4.20}$$

which is equivalent to say that

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(f)(\omega) = M_\infty(\omega) \int_X f h\, dm \qquad \text{for every } \omega \in \mathbf{\Omega}_0 \cap \{M_\infty > 0\}. \tag{4.21}$$

Since, for every function f on X such that $|f|$ is bounded by ch for some $c > 0$,

$$e^{\lambda_1 t} |Z_t(f)| \leq e^{\lambda_1 t} Z_t(|f|) \leq c M_t,$$

(4.21) holds automatically on $\{M_\infty = 0\}$. This completes the proof of the theorem. □

In a similar way to that yielding Corollary 4.5, we obtain from Theorem 4.7 and Lemma 4.2 the following:

Corollary 4.10. *Let* $\mathbf{\Omega}_0$ *be the same as in Theorem* 4.7*.*
(i) (A law of large numbers) *It holds that*

$$\lim_{t\to\infty} \frac{Z_t(A)(\omega)}{\mathbf{E}_x\left[Z_t(A)\right]} = \frac{M_\infty(\omega)}{h(x)}$$

for every $\omega \in \mathbf{\Omega}_0$ *and for every relatively compact Borel subset* A *in* X *having* $m(A) > 0$ *and* $m(\partial A) = 0$.
(ii) *Suppose also that Assumption* 4.4 *holds. Then*

$$\lim_{t\to\infty} \frac{Z_t(A)(\omega)}{Z_t(B)(\omega)} = \frac{\int_A h\, dm}{\int_B h\, dm}$$

for every $\omega \in \mathbf{\Omega}_0 \cap \{e_0 = \infty\}$ *and for every pair of relatively compact Borel subsets* A *and* B *in* X *having* $m(A), m(B) > 0$ *and* $m(\partial A) = m(\partial B) = 0$ *respectively.*

4.3 Examples

We apply the results above to branching Brownian motions and branching symmetric α-stable processes. Let $\mathbf{M}^\alpha = (X_t, P_x)$ be the symmetric α-stable process on $\mathbb{R}^d$ and $(\mathcal{E}^\alpha, \mathcal{F}^\alpha)$ the associated Dirichlet form on $L^2(\mathbb{R}^d)$. Denote by $\mathbf{M}^D = (X_t^D, P_x^D)$ the absorbing symmetric α-stable process on an open set D in $\mathbb{R}^d$ and by $(\mathcal{E}^D, \mathcal{F}^D)$ the associated Dirichlet form. Let

$G^D(x,y)$ and $G^D_\beta(x,y)$ be the Green function and the β-resolvent density of $\mathbf{M}^D$ respectively. We set

$$\lambda_1(\mu; D) = \inf\left\{\mathcal{E}^D(u,u) - \int_D u^2\, d\mu : u \in C_0^\infty(D), \quad \int_D u^2\, dx = 1\right\}.$$

Let $\overline{\mathbf{M}^D} = (\mathbf{X}_t^D, \mathbf{P}_x)$ be the branching symmetric α-stable process with motion component $\mathbf{M}^D$ and branching rate measure $\mu \in \mathcal{K}_\infty^D$. Let Q be the same as before. Suppose that $\lambda_1 := \lambda_1((Q-1)\mu; D) < 0$. If D is bounded, then Proposition 4.3, Corollary 4.5, Theorem 4.7 and Corollary 4.10 hold. On the other hand, if $D = \mathbb{R}^d$, then Proposition 4.3, Corollary 4.5 (i) and (ii), Theorem 4.7 and Corollary 4.10 hold. Otherwise, Proposition 4.3, Theorem 4.7 and Corollary 4.10 hold.

From now on, we use the following notation: for functions f and g on a space E and a subset $F \subset E$, we write $f \approx g$ on F, if there exist positive constants $c_1 > c_2 > 0$ such that for any $x \in F$,

$$c_2 g(x) \leq f(x) \leq c_1 g(x).$$

Example 4.11. Suppose that $d = 1$ and $1 < \alpha \leq 2$. Let $D = (-R, R)$ for $R > 0$ and $a \in D$. Let $\lambda_1 = \lambda_1(\delta_a; (-R,R))$, where δ_a is the Dirac measure at a. We denote by $\overline{\mathbf{M}^D} = (\mathbf{X}_t^D, \mathbf{P}_x)$ the binary branching symmetric α-stable process with motion component $\mathbf{M}^D$ and branching rate δ_0. We first suppose that $\alpha = 2$. Note that we can calculate λ_1 and the corresponding ground state in Example 6.8 below. Hence, if $R > 0$ satisfies the right hand side of (3.23), then for any $r \in (a, R)$ and $\delta > 0$, we have by Proposition 4.3 (ii) $\mathbf{P}_x$-a.s.

$$\begin{aligned}&\lim_{n\to\infty} e^{\lambda_1 n\delta} Z_{n\delta}((-r,r)^c)\\ &\quad= \frac{C_1}{\sqrt{-2\lambda_1}}\left(\sinh\{2\sqrt{-2\lambda_1}(R-a)\} + \sinh\{2\sqrt{-2\lambda_1}(R+a)\}\right)\sinh^2\{\sqrt{-2\lambda_1}(R-r)\} M_\infty,\end{aligned}$$

and by Corollary 4.5 (iii)

$$\mathbf{P}_x\left(\lim_{n\to\infty} \frac{Z_{n\delta}((-r,r)^c)}{Z_{n\delta}} = C_2 \sinh^2\{\sqrt{-2\lambda}(R-r)\} \,\middle|\, e_0 = \infty\right) = 1,$$

where $C_1 = C_1(a, R, \lambda_1)$ is the positive constant which will be defined in (6.6) below and

$$C_2 = C_2(a, R, \lambda_1) = \frac{\sinh\{2\sqrt{-2\lambda_1}(R-a)\} + \sinh\{2\sqrt{-2\lambda_1}(R+a)\}}{2\sinh\{2\sqrt{-2\lambda_1}R\}\sinh\{\sqrt{-2\lambda_1}(R-a)\}\sinh\{\sqrt{-2\lambda_1}(R+a)\}}.$$

We also have by Theorem 4.7 $\mathbf{P}_x$-a.s.

$$\begin{aligned}\lim_{t\to\infty} e^{\lambda_1 t} Z_t((-r,r)) =& \frac{C_1}{\sqrt{-2\lambda_1}}\left(\sinh\{2\sqrt{-2\lambda_1}(R-a)\} + \sinh\{2\sqrt{-2\lambda_1}(R+a)\}\right)\\ &\left(\sinh^2\{\sqrt{-2\lambda_1}R\} - \sinh^2\{\sqrt{-2\lambda_1}(R-r)\}\right) M_\infty\end{aligned}$$

for any $r \in (a, R)$, and by Corollary 4.10 (i) $\mathbf{P}_x$-a.s.

$$\begin{aligned}&\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x\left[Z_t(A)\right]}\\ &= \begin{cases} \dfrac{1}{C_1 \sinh\{2\sqrt{-2\lambda_1}(R-a)\}\sinh\{2\sqrt{-2\lambda_1}(R+x)\}} M_\infty, & x \in (-R, a] \\ \dfrac{1}{C_1 \sinh\{2\sqrt{-2\lambda_1}(R+a)\}\sinh\{2\sqrt{-2\lambda_1}(R-x)\}} M_\infty, & x \in (a, R) \end{cases}\end{aligned}$$

for every relatively compact Borel subset A in $(-R, R)$ whose boundary has zero Lebesgue measure. We next suppose that $1 < \alpha < 2$. Note that we already obtain the decay rate of the ground state by (3.24). Hence, if $R > 0$ satisfies the right hand side of (3.23), then for any $r \in (a, R)$ and $\delta > 0$, we have by Proposition 4.3 (ii) $\mathbf{P}_x$-a.s.

$$\lim_{n\to\infty} e^{\lambda_1 n\delta} Z_{n\delta}((-r,r)^c) = O\left((R-r)^{(\alpha+2)/2}\right),$$

and by Corollary 4.5

$$\mathbf{P}_x\left(\lim_{n\to\infty} \frac{Z_{n\delta}((-r,r)^c)}{Z_{n\delta}} = O\left((R-r)^{(\alpha+2)/2}\right) \,\middle|\, e_0 = \infty\right) = 1.$$

We also obtain by Theorem 4.7 $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t((-r,r)) = \left(\int_{-\infty}^{\infty} h\,dx - O\left((R-r)^{(\alpha+2)/2}\right)\right) M_\infty$$

for any $r \in (a, R)$, and by Corollary 4.10 (i) $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x\left[Z_t(A)\right]} \approx \begin{cases} (R+x)^{-\alpha/2} M_\infty & \text{as } x \to -R \\ (R-x)^{-\alpha/2} M_\infty & \text{as } x \to R \end{cases}$$

for every relatively compact Borel subset A in $(-R, R)$ whose boundary has zero Lebesgue measure.

Example 4.12. Suppose that $d = 1$ and $1 < \alpha \le 2$. Let $\lambda_1(\alpha) = \lambda_1(\delta_0; \mathbb{R})$ and denote by h the corresponding ground state. We can then obtain $\lambda_1(\alpha)$ and h explicitly in Example 6.10 below, where $Q = 2$. We can also see the decay rate of h at infinity by (1.25) and (1.26).

Let $\overline{\mathbf{M}^\alpha} = (\mathbf{X}_t, \mathbf{P}_x)$ be the branching symmetric α-stable process with motion component $\mathbf{M}^\alpha$ and branching rate δ_0. If $\alpha = 2$, then for any $r > 0$, we get by Proposition 4.3 (i) in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} e^{-t/2} Z_t((-r,r)^c) = 2e^{-r} M_\infty,$$

and by Corollary 4.5 (ii) in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} \frac{Z_t((-r,r)^c)}{Z_t} = e^{-r}.$$

We also get by Theorem 4.7 $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{-t/2} Z_t((-r,r)) = 2(1 - e^{-r}) M_\infty$$

for large $r > 0$, and by Corollary 4.10 (i) $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x\left[Z_t(A)\right]} = e^{|x|} M_\infty$$

for every $x \in \mathbb{R}$ and for every relatively compact Borel subset A in $\mathbb{R}$ whose boundary has zero Lebesgue measure. On the other hand, if $1 < \alpha < 2$, then for large $r > 0$, we have by Proposition 4.3 (i) in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} e^{\lambda_1(\alpha) t} Z_t((-r,r)^c) = O\left(r^{-\alpha}\right) M_\infty,$$

and by Corollary 4.5 (ii) and (1.26) in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} \frac{Z_t((-r,r)^c)}{Z_t} = O\left(r^{-\alpha}\right).$$

We also have by Theorem 4.7 $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{\lambda_1(\alpha)t} Z_t((-r,r)) = \left(\int_{-\infty}^{\infty} h(x)\,dx - O\left(r^{-\alpha}\right)\right) M_\infty$$

for large $r > 0$, and by Corollary 4.10 $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x\left[Z_t(A)\right]} \approx |x|^{1+\alpha} M_\infty \quad \text{as} \quad |x| \to \infty$$

for every relatively compact Borel subset A in $\mathbb{R}$ whose boundary has zero Lebesgue measure.

Example 4.13. Suppose that $1 < \alpha \leq 2$ and $d > \alpha$. Let δ_R be the surface measure on $\partial B_R = \{x \in \mathbb{R}^d : |x| = R\}$ and $\lambda_1 := \lambda_1(\delta_R; \mathbb{R}^d)$. Denote by $\overline{\mathbf{M}^\alpha} = (\mathbf{X}_t, \mathbf{P}_x)$ the binary branching symmetric α-stable process with motion component $\mathbf{M}^\alpha$ and branching rate δ_R. Assume that the radius $R > 0$ satisfies the right hand side of (3.26). Denote by $B(r)$ the open ball with radius $r > 0$ and centered at the origin, $B(r) = \{x \in \mathbb{R}^d : |x| < r\}$. If $\alpha = 2$, then for large $r > 0$, we have by Proposition 4.3 (i) and Remark 4.6 in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(B(r)^c) = o\left(e^{-\sqrt{-2\lambda_1}r}\right) M_\infty,$$

and by Corollary 4.5 (ii), Remark 4.6 and (1.25) in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} \frac{Z_t(B(r)^c)}{Z_t} = o\left(e^{-\sqrt{-2\lambda_1}r}\right).$$

We also obtain by Theorem 4.7 $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(B(r)) = \left(\int_{\mathbb{R}^d} h\,dx - o\left(e^{-\sqrt{-2\lambda_1}r}\right)\right) M_\infty$$

for large $r > 0$, and by Corollary 4.10 (i) $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} \frac{Z_t(A)}{\mathbf{E}_x\left[Z_t(A)\right]} = o\left(e^{\sqrt{-2\lambda_1}|x|}\right) M_\infty \quad \text{as} \quad |x| \to \infty$$

for every relatively compact Borel subset A in $\mathbb{R}^d$ whose boundary has zero Lebesgue measure. On the other hand, if $1 < \alpha < 2$, then for large $r > 0$, we get by Proposition 4.3 (i) and Remark 4.6 in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(B(r)^c) = O\left(r^{-\alpha}\right) M_\infty,$$

and by Corollary 4.5 (ii), Remark 4.6 and (1.26) in $\mathbf{P}_x$-probability

$$\lim_{t\to\infty} \frac{Z_t(B(r)^c)}{Z_t} = O\left(r^{-\alpha}\right).$$

We also get by Theorem 4.7 $\mathbf{P}_x$-a.s.

$$\lim_{t\to\infty} e^{\lambda_1 t} Z_t(B(r)) = \left(\int_{\mathbb{R}^d} h\,dx - O\left(r^{-\alpha}\right)\right) M_\infty$$

for large $r > 0$, and by Corollary 4.10 (i) $\mathbf{P}_x$-a.s.

$$\lim_{t \to \infty} \frac{Z_t(A)}{\mathbf{E}_x\left[Z_t(A)\right]} \approx |x|^{d+\alpha} M_\infty \quad \text{as} \quad |x| \to \infty$$

for every relatively compact subset A in $\mathbb{R}^d$ whose boundary has zero Lebesgue measure.

Chapter 5

Variational formula for Dirichlet forms and its applications

In this chapter, we prove a variational formula for Dirichlet forms generated by general symmetric Markov processes. As its applications, we obtain lower bound estimates of the bottoms of the spectrum for symmetric Markov processes. Moreover, for a positive measure μ charging no set of zero capacity, we give a new proof of an $L^2(\mu)$-estimate of functions in Dirichlet spaces.

5.1 Variational formula for Dirichlet forms

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. In [24] Donsker-Varadhan proved a large deviation principle of occupation distributions of conservative Markov processes on X with the so-called *I-function* as its rate function: let $\mathcal{L}$ be the Feller generator of a Markov process and $\mathcal{D}(\mathcal{L})$ the domain of $\mathcal{L}$. Then the I-function is defined by

$$I(\mu) = -\inf_{u \in \mathcal{D}^{++}(\mathcal{L})} \int_X \frac{\mathcal{L}u}{u}\, d\mu, \quad \mu \in \mathcal{P}(X), \tag{5.1}$$

where $\mathcal{D}^{++}(\mathcal{L}) = \{u \in \mathcal{D}(\mathcal{L}) : \inf_{x \in X} u(x) > 0\}$. Moreover, if the Markov process is m-symmetric, then they identified the I-function with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ as follows:

$$I(\mu) = \begin{cases} \mathcal{E}(f,f), & \text{if } f = \sqrt{\dfrac{d\mu}{dm}} \in \mathcal{F}, \\ \infty, & \text{otherwise} \end{cases}$$

([24, Theorem 5]). In other words, the Dirichlet form is expressed as

$$\mathcal{E}(f,f) = -\inf_{u \in \mathcal{D}^{++}(\mathcal{L})} \int_X \frac{\mathcal{L}u}{u} f^2\, dm, \qquad f \in \mathcal{F}. \tag{5.2}$$

Here we call the relation like (5.2) the *variational formula for the Dirichlet form* $(\mathcal{E}, \mathcal{F})$. In this section, we extend this formula to general symmetric Markov processes with jumps and killings.

Let $\mathbf{M} = (X_t, P_x)$ be an m-symmetric Markov process on X with right continuous sample paths. Denote by $\mathcal{B}^*(X)$ the collection of *universally measurable* subsets,

$$\mathcal{B}^*(X) = \bigcap_{\mu \in \mathcal{P}(X)} \mathcal{B}^\mu(X),$$

where $\mathcal{P}(X)$ is the set of probability measures on X and $\mathcal{B}^{\mu}(X)$ is the completion of Borel sets $\mathcal{B}(X)$ with respect to the measure $\mu \in \mathcal{P}(X)$. A function $f \in \mathcal{B}^*(X)$ is then said to be *finely continuous*, if

$$P_x\left(f(X_t) \text{ is right continuous with respect to } t \in [0,\infty)\right) = 1 \quad \text{for any } x \in X.$$

Then any continuous function is finely continuous by the right continuity of sample paths. Denote by $C^{\varphi}(X)$ the set of finely continuous functions on X. Let $C_b^{\varphi}(X)$ be the set of bounded finely continuous functions on X and $C_b^{\varphi,+}(X)$ the set of nonnegative functions in $C_b^{\varphi}(X)$. Denote by p_t the Markovian transition semigroup of $\mathbf{M}$, $p_t f(x) = E_x[f(X_t)],\ f \in \mathcal{B}^*(X)$. We now define the extended generator of $\mathbf{M}$ as follows:

Definition 5.1. *Let*

$$\mathcal{D}\left(\hat{\mathcal{L}}\right) = \left\{ u \in C_b^{\varphi}(X) : {}^{\exists} g \in C_b^{\varphi}(X) \text{ s.t. } p_t u = u + \int_0^t p_s g\, ds,\ \forall t > 0 \right\}.$$

Then g is uniquely determined for each $u \in \mathcal{D}\left(\hat{\mathcal{L}}\right)$. The function g is denoted by $\hat{\mathcal{L}}u$ and $\hat{\mathcal{L}}$ is called the extended generator of $\mathbf{M}$.

We learn the notion of extended generators from [63]. Note that, the function $u \in C_b^{\varphi}(X)$ belongs to $\mathcal{D}\left(\hat{\mathcal{L}}\right)$ if and only if there exists a function $g \in C_b^{\varphi}(X)$ such that $M_t = u(X_t) - u(X_0) - \int_0^t g(X_s)\, ds$ is a martingale. Moreover, we have

Lemma 5.2. ([63, §3 Theorem 1.2]) *For any extended generator $\hat{\mathcal{L}}$, it holds that*

$$\mathcal{D}\left(\hat{\mathcal{L}}\right) = G_\alpha(C_b^{\varphi}(X)), \quad \alpha > 0.$$

Proof. Take $u \in \mathcal{D}\left(\hat{\mathcal{L}}\right)$ and let $v = \hat{\mathcal{L}}u$. Then by definition,

$$p_t u = u + \int_0^t p_s v\, ds. \tag{5.3}$$

Since

$$\begin{aligned} \int_0^\infty e^{-\alpha t}\left(\int_0^t p_s v\, ds\right) dt &= \int_0^\infty \left(\int_s^\infty e^{-\alpha t}\, dt\right) p_s v\, ds \\ &= \frac{1}{\alpha}\int_0^\infty e^{-\alpha s} p_s v\, ds = \frac{1}{\alpha} G_\alpha v \end{aligned}$$

by Fubini's theorem, we see from (5.3) that

$$G_\alpha u = \frac{1}{\alpha}u + \frac{1}{\alpha}G_\alpha v.$$

Thus $u = G_\alpha(\alpha u - v) \in G_\alpha(C_b^{\varphi}(X))$, which yields that $\mathcal{D}\left(\hat{\mathcal{L}}\right) \subset G_\alpha(C_b^{\varphi}(X))$.

Suppose that $f \in C_b^{\varphi}(X)$. Then

$$\begin{aligned} p_t G_\alpha f - G_\alpha f &= \int_0^\infty e^{-\alpha s} p_{t+s} f\, ds - \int_0^\infty e^{-\alpha s} p_s f\, ds \\ &= e^{\alpha t}\int_t^\infty e^{-\alpha s} p_s f\, ds - \int_0^\infty e^{-\alpha s} p_s f\, ds. \end{aligned}$$

Since

$$\frac{d}{dt}\left(p_t G_\alpha f - G_\alpha f\right) = \alpha e^{\alpha t}\int_t^\infty e^{-\alpha s} p_s f\, ds - p_t f$$
$$= p_t(\alpha G_\alpha f - f),$$

it holds that

$$p_t G_\alpha f - G_\alpha f = \int_0^t p_s\left(\alpha G_\alpha f - f\right) ds.$$

Noting that $G_\alpha(C_b^\varphi(X)) \subset C_b^\varphi(X)$ by [9, §2], we have $\alpha G_\alpha f - f \in C_b^\varphi(X)$. Hence $G_\alpha f \in \mathcal{D}\left(\hat{\mathcal{L}}\right)$ and $\hat{\mathcal{L}} G_\alpha f = \alpha G_\alpha f - f$, which imply that $G_\alpha(C_b^\varphi(X)) \subset \mathcal{D}\left(\hat{\mathcal{L}}\right)$. □

Let $\mathcal{B}_b^*(X)$ denote the set of bounded $\mathcal{B}^*(X)$-measurable functions on X and $\mathcal{B}_b^{*,+}(X)$ the set of nonnegative functions in $\mathcal{B}_b^*(X)$. Let $\mathcal{D}^+\left(\hat{\mathcal{L}}\right)$ be the set of nonnegative functions in $\mathcal{D}\left(\hat{\mathcal{L}}\right)$. Taking account of killings or explosions, we define the function $\hat{I}$ on $\mathcal{P}(X)$ by

$$\hat{I}(\mu) = -\inf_{u\in\mathcal{D}^+\left(\hat{\mathcal{L}}\right),\, \varepsilon>0}\int_X \frac{\hat{\mathcal{L}}u}{u+\varepsilon}\, d\mu. \tag{5.4}$$

The function $\hat{I}$ is a modification of the *I-function* defined in (5.1). By adding $\varepsilon > 0$ on the denominator of the integrand of $\hat{I}$, the integral on the right hand side of (5.4) is finite for any $u \in \mathcal{D}^+(\hat{\mathcal{L}})$. We also define the function I_α on $\mathcal{P}(X)$ by

$$I_\alpha(\mu) = -\inf_{u\in\mathcal{B}_b^{*,+}(X),\varepsilon>0}\int_X \log\left(\frac{\alpha G_\alpha u + \varepsilon}{u+\varepsilon}\right) d\mu, \quad \alpha > 0.$$

Next lemma is a resolvent version of Donsker and Varadhan [24, Lemma 3.1].

Lemma 5.3. *It holds that*

$$\alpha I_\alpha(\mu) \le \hat{I}(\mu)$$

for any $\mu \in \mathcal{P}(X)$ and $\alpha > 0$.

Proof. For $f \in C_b^{\varphi,+}(X)$ and $\beta > 0$, let $u = \beta G_\beta f \in \mathcal{D}^+\left(\hat{\mathcal{L}}\right)$. Put

$$\phi(\alpha) = -\int_X \log\left(\frac{\alpha G_\alpha u + \varepsilon}{u+\varepsilon}\right) d\mu, \quad \varepsilon > 0$$

for $\alpha > 0$. Since

$$\alpha G_\alpha^2 u - G_\alpha u = \hat{\mathcal{L}} G_\alpha^2 u$$

as showed in the proof of Lemma 5.2 and

$$\frac{d}{d\alpha} G_\alpha u = -G_\alpha^2 u,$$

it follows that

$$\frac{d\phi}{d\alpha}(\alpha) = \int_X \frac{\hat{\mathcal{L}} G_\alpha^2 u}{\alpha G_\alpha u + \varepsilon} d\mu. \tag{5.5}$$

As a direct calculation yields that

$$\begin{aligned}\frac{\hat{\mathcal{L}}G_\alpha^2u}{\alpha G_\alpha u+\varepsilon}-\frac{\hat{\mathcal{L}}G_\alpha^2u}{\alpha^2G_\alpha^2u+\varepsilon}&=\frac{\alpha^2G_\alpha^2u+\varepsilon-(\alpha G_\alpha u+\varepsilon)}{(\alpha G_\alpha u+\varepsilon)(\alpha^2G_\alpha^2u+\varepsilon)}\hat{\mathcal{L}}G_\alpha^2u\\&=\frac{\left(\hat{\mathcal{L}}G_\alpha^2u\right)^2}{(\alpha G_\alpha u+\varepsilon)(\alpha^2G_\alpha^2u+\varepsilon)}\ge 0,\end{aligned}$$

it holds that

$$\begin{aligned}\int_X\frac{\hat{\mathcal{L}}G_\alpha^2u}{\alpha G_\alpha u+\varepsilon}\,d\mu&\ge\int_X\frac{\hat{\mathcal{L}}G_\alpha^2u}{\alpha^2G_\alpha^2u+\varepsilon}\,d\mu\\&=-\frac{1}{\alpha^2}\int_X\frac{-\hat{\mathcal{L}}G_\alpha^2u}{G_\alpha^2u+\varepsilon/\alpha^2}\,d\mu\\&\ge-\frac{1}{\alpha^2}\hat{I}(\mu).\end{aligned}$$

By integrating both sides of (5.5) by α,

$$-\phi(\alpha)=\int_\alpha^\infty\phi'(\beta)\,d\beta\ge-\frac{1}{\alpha}\hat{I}(\mu).$$

Therefore,

$$\frac{\hat{I}(\mu)}{\alpha}\ge-\inf_{u\in\mathcal{D}^+(\hat{\mathcal{L}}),\varepsilon>0}\int_X\log\left(\frac{\alpha G_\alpha u+\varepsilon}{u+\varepsilon}\right)\,d\mu.$$

Since $G_\alpha(C_b^{\varphi,+}(X))\subset\mathcal{D}^+\left(\hat{\mathcal{L}}\right)$ by Lemma 5.2, it follows that $\alpha G_\alpha f\in\mathcal{D}^+\left(\hat{\mathcal{L}}\right)$ for any $f\in C_b^{\varphi,+}(X)$. Furthermore, it holds that

$$\frac{\alpha G_\alpha(\beta G_\beta f)+\varepsilon}{\beta G_\beta f+\varepsilon}=\frac{\beta G_\beta(\alpha G_\alpha f)+\varepsilon}{\beta G_\beta f+\varepsilon}\to\frac{\alpha G_\alpha f+\varepsilon}{f+\varepsilon}\quad\text{as}\quad\beta\to\infty$$

because the fine continuity of f implies that

$$\begin{aligned}\lim_{\alpha\to\infty}\alpha G_\alpha f(x)&=\lim_{\alpha\to\infty}\alpha E_x\left[\int_0^\infty e^{-\alpha u}f(X_u)\,du\right]\\&=\lim_{\alpha\to\infty}E_x\left[\int_0^\infty e^{-u}f(X_{u/\alpha})\,du\right]=f(x).\end{aligned}$$

Thus

$$-\inf_{u\in\mathcal{D}^+(\hat{\mathcal{L}}),\varepsilon>0}\int_X\log\left(\frac{\alpha G_\alpha u+\varepsilon}{u+\varepsilon}\right)\,d\mu=-\inf_{u\in C_b^{\varphi,+}(X),\varepsilon>0}\int_X\log\left(\frac{\alpha G_\alpha u+\varepsilon}{u+\varepsilon}\right)\,d\mu.$$

Define the measure μ_α on $\mathcal{B}^*(X)$ by

$$\mu_\alpha(A)=\int_X\alpha G_\alpha(x,A)\,\mu(dx),\quad A\in\mathcal{B}^*(X).$$

Take a sequence $\{g_n\}\subset C_b^{\varphi,+}(X)$ such that

$$\lim_{n\to\infty}\int_X|g_n-v|d(\mu_\alpha+\mu)=0$$

for each $v \in \mathcal{B}_b^{*,+}(X)$. Then

$$\begin{aligned}\int_X |\alpha G_\alpha g_n - \alpha G_\alpha v|\, d\mu &\leq \int_X \alpha G_\alpha |g_n - v|\, d\mu \\ &= \int_X |g_n - v|\, d\mu_\alpha \to 0, \quad n \to \infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \int_X \log\left(\frac{\alpha G_\alpha g_n + \varepsilon}{g_n + \varepsilon}\right) d\mu = \int_X \log\left(\frac{\alpha G_\alpha v + \varepsilon}{v + \varepsilon}\right) d\mu.$$

As a result, we obtain

$$\begin{aligned}-\inf_{u \in C_b^{\varphi,+}(X), \varepsilon>0} \int_X \log\left(\frac{\alpha G_\alpha u + \varepsilon}{u + \varepsilon}\right) d\mu &= -\inf_{u \in \mathcal{B}_b^{*,+}(X), \varepsilon>0} \int_X \log\left(\frac{\alpha G_\alpha u + \varepsilon}{u + \varepsilon}\right) d\mu \\ &= I_\alpha(\mu),\end{aligned}$$

which completes the proof. $\square$

Theorem 5.4. *Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with $\mathbf{M}$. Then it holds that*

$$\mathcal{E}(f,f) = -\inf_{u \in \mathcal{D}^+(\hat{\mathcal{L}}),\, \varepsilon>0} \int_X \frac{\hat{\mathcal{L}} u}{u + \varepsilon} f^2 dm \tag{5.6}$$

for any $f \in \mathcal{F}$ with $f \geq 0$ m-a.e. on X.

Proof. Let $f \in \mathcal{F}$ with $f \geq 0$ m-a.e. on X and $f_n = f \wedge n$. Since $x < -\log(1-x)$ for all $x \in (-\infty, 1)$ and

$$-\infty < \frac{f_n - \alpha G_\alpha f_n}{f_n + \varepsilon} < 1,$$

we have

$$\int_X \frac{f_n - \alpha G_\alpha f_n}{f_n + \varepsilon} f^2\, dm \leq -\int_X \log\left(\frac{\alpha G_\alpha f_n + \varepsilon}{f_n + \varepsilon}\right) f^2\, dm.$$

By Lemma 5.3, the right hand side above is not greater than

$$I_\alpha(f^2 \cdot m) \leq \frac{\hat{I}(f^2 \cdot m)}{\alpha}.$$

Since

$$\begin{aligned}\left|\frac{f_n - \alpha G_\alpha f_n}{f_n + \varepsilon}\right| f^2 &= \left|\frac{n - \alpha G_\alpha f_n}{n + \varepsilon}\right| f^2 \mathbf{1}_{\{f>n\}} + \left|\frac{f - \alpha G_\alpha f_n}{f + \varepsilon}\right| f^2 \mathbf{1}_{\{f\leq n\}} \\ &\leq \frac{n + \alpha G_\alpha n}{n} f^2 + (f + \alpha G_\alpha f) f \\ &\leq f\,(3f + \alpha G_\alpha f) \in L^1(X; m),\end{aligned}$$

we obtain, by letting first $n \to \infty$ and then $\varepsilon \to 0$,

$$\int_X \alpha\,(f - \alpha G_\alpha f)\, f\, dm \leq \hat{I}(f^2 \cdot m).$$

Noting that $\lim_{\alpha\to\infty}\int_X \alpha\,(f-\alpha G_\alpha f)\,f\,dm = \mathcal{E}(f,f)$ by [29, Lemma 1.3.4], we get $\mathcal{E}(f,f) \le \hat{I}(f^2\cdot m)$.

For any $\phi \in \mathcal{D}^+\left(\hat{\mathcal{L}}\right)$, put

$$\hat{p}_t^{\phi} f(x) = E_x\left[\frac{\phi(X_t)+\varepsilon}{\phi(X_0)+\varepsilon}\exp\left(-\int_0^t \frac{\hat{\mathcal{L}}\phi}{\phi+\varepsilon}(X_s)\,ds\right) f(X_t)\right].$$

Then it follows in a similar way to [51] and [52] that $\hat{p}_t^{\phi}$ is $(\phi+\varepsilon)^2 m$-symmetric and $\hat{p}_t^{\phi} 1 \le 1$. Define

$$\hat{S}_t^{\phi} f(x) = E_x\left[\exp\left(-\int_0^t \frac{\hat{\mathcal{L}}\phi}{\phi+\varepsilon}(X_s)\,ds\right) f(X_t)\right].$$

Since

$$\hat{S}_t^{\phi} f(x) = (\phi(x)+\varepsilon)\hat{p}_t^{\phi}\left(\frac{f}{\phi+\varepsilon}\right)(x),$$

Schwarz's inequality yields that

$$\begin{aligned}
\int_X \left(\hat{S}_t^{\phi} f\right)^2 dm &= \int_X \left\{\hat{p}_t^{\phi}\left(\frac{f}{\phi+\varepsilon}\right)\right\}^2 (\phi+\varepsilon)^2 dm \\
&\le \int_X \hat{p}_t^{\phi} 1 \hat{p}_t^{\phi}\left\{\left(\frac{f}{\phi+\varepsilon}\right)^2\right\} (\phi+\varepsilon)^2 dm \\
&\le \int_X \hat{p}_t^{\phi}\left\{\left(\frac{f}{\phi+\varepsilon}\right)^2\right\} (\phi+\varepsilon)^2 dm.
\end{aligned}$$

By the $(\phi+\varepsilon)^2 m$-symmetry of $\hat{p}_t^{\phi}$, the last term above is equal to

$$\int_X \hat{p}_t^{\phi} 1 \left(\frac{f}{\phi+\varepsilon}\right)^2 (\phi+\varepsilon)^2 dm \le \int_X f^2\,dm.$$

Because $\hat{\mathcal{L}}\phi/(\phi+\varepsilon)$ is bounded, we see from the Feynman-Kac formula ([1, Theorem 4.1]) that

$$\begin{aligned}
0 &\le \lim_{t\to 0}\frac{1}{t}\int_X \left(f-\hat{S}_t^{\phi} f\right) f\,dm \\
&= \mathcal{E}(f,f) + \int_X \frac{\hat{\mathcal{L}}\phi}{\phi+\varepsilon} f^2\,dm,
\end{aligned} \tag{5.7}$$

which implies that $\mathcal{E}(f,f) \ge \hat{I}(f^2\cdot m)$ and the equation (5.6) follows. $\square$

Remark 5.5. In [24], Donsker and Varadhan assumed that the Markov process satisfies the Feller property, that is, $p_t(C_b(X)) \subset C_b(X)$. Here $C_b(X)$ stands for the set of bounded continuous functions on X. In [52], the argument of [24] was modified by using the α-resolvent under the assumption that $\mathbf{M}$ satisfies the strong Feller property, $G_\alpha(\mathcal{B}_b(X)) \subset C_b(X)$. One of our main objectives is to obtain the lower bounds of the principal eigenvalues for time changed processes; however, it is difficult in general to prove the Feller property of time changed processes (see [47], where one-dimensional diffusion processes are discussed). Here we would like to emphasize that it always holds that $G_\alpha(C_b^{\varphi}(X)) \subset C_b^{\varphi}(X)$ as mentioned in the proof of Lemma 5.2. This is the reason we modify the I-function by the α-resolvent and use the notion of extended generators in our argument.

Remark 5.6. Suppose that $(\mathcal{E}, \mathcal{F})$ is a local Dirichlet form. Then the corresponding process is an m-symmetric diffusion process on X. Define

$$\mathcal{D}_{loc}\left(\hat{\mathcal{L}}\right) = \left\{u \in C^{\varphi}(X) : {}^{\exists}g \in C^{\varphi}(X) \text{ s.t. } u(X_t) - u(X_0) - \int_0^t g(X_s)\, ds \text{ is a local martingale,} \right.$$
$$\left. \forall t < \zeta \text{ and } \frac{g}{u+\varepsilon} \in \mathcal{B}_b^*(X)\right\}.$$

For $u \in \mathcal{D}_{loc}\left(\hat{\mathcal{L}}\right)$, we denote by $\hat{\mathcal{L}}u$ the function g in the definition of $\mathcal{D}_{loc}\left(\hat{\mathcal{L}}\right)$. Let $\mathcal{D}_{loc}^{+}\left(\hat{\mathcal{L}}\right)$ be the set of nonnegative functions in $\mathcal{D}_{loc}\left(\hat{\mathcal{L}}\right)$. We then have

$$\mathcal{E}(f,f) = -\inf_{u \in \mathcal{D}_{loc}^{+}(\hat{\mathcal{L}}),\, \varepsilon>0} \int_X \frac{\hat{\mathcal{L}}u}{u+\varepsilon} f^2\, dm. \tag{5.8}$$

Indeed, the upper estimate of $\mathcal{E}(f,f)$ is clear from (5.6) because $\mathcal{D}^{+}\left(\hat{\mathcal{L}}\right) \subset \mathcal{D}_{loc}^{+}\left(\hat{\mathcal{L}}\right)$. Since $\hat{p}_t^{\phi}1 \le 1$ for $\phi \in \mathcal{D}_{loc}^{+}\left(\hat{\mathcal{L}}\right)$ and $\hat{\mathcal{L}}\phi/(\phi+\varepsilon)$ is bounded, the lower estimate follows by the same argument as Theorem 5.4. Hence we can take unbounded functions as test functions in the right hand side of (5.8). For instance, let us consider the Ornstein-Uhlenbeck process on $(0,\infty)$ absorbed at 0. Then $u(x) = x \in \mathcal{D}_{loc}^{+}\left(\hat{\mathcal{L}}\right)$ and

$$\hat{\mathcal{L}}u(x) = \frac{d^2u}{dx^2}(x) - x\frac{du}{dx}(x) = -x.$$

5.2 Applications

In this section, we assume that $\mathbf{M}$ is transient. Let λ_0 be the bottom of the spectrum of the Dirichlet form $(\mathcal{E}, \mathcal{F})$:

$$\lambda_0 = \inf\left\{\mathcal{E}(f,f) : f \in \mathcal{F}, \int_X f^2\, dm = 1\right\}.$$

On account of Theorem 5.4 we have

Theorem 5.7. *It holds that*

$$\lambda_0 = \inf\left\{\sup_{u \in \mathcal{D}^{+}(\hat{\mathcal{L}}),\, \varepsilon>0} \int_X \frac{-\hat{\mathcal{L}}u}{u+\varepsilon} f^2 dm : f \in \mathcal{F}, \int_X f^2\, dm = 1\right\}. \tag{5.9}$$

Proof. Since $\mathcal{E}(f,f) \ge \mathcal{E}(|f|,|f|)$ for $f \in \mathcal{F}$ ([29, p.5]), it holds that

$$\lambda_0 = \inf\left\{\mathcal{E}(|f|,|f|) : f \in \mathcal{F}, \int_X f^2\, dm = 1\right\}.$$

By applying Theorem 5.4 to $\mathcal{E}(|f|,|f|)$, the equation (5.9) follows. □

We now derive the *generalized Barta's inequality*, a lower bound estimate of λ_0. Set

$$\mathcal{C} = \left\{u \in \mathcal{D}^{+}\left(\hat{\mathcal{L}}\right) : u > 0,\, -\hat{\mathcal{L}}u > 0\right\}.$$

We then have

Theorem 5.8. (Generalized Barta's inequality) *It holds that*

$$\lambda_0 \geq \inf_{x \in X} \left(-\frac{\hat{\mathcal{L}} u}{u} \right)(x) \tag{5.10}$$

for any $u \in \mathcal{C}$. In particular, if there exist $u \in \mathcal{C}$ and $\kappa > 0$ such that $-\hat{\mathcal{L}} u \geq \kappa u$, then $\lambda_0 \geq \kappa$.

Proof. Let $u \in \mathcal{C}$. From Theorem 5.4 and Fatou's lemma, it follows that, for any $f \in \mathcal{F}$ with $\int_X f^2 \, dm = 1$,

$$\begin{aligned} \mathcal{E}(f, f) &\geq \liminf_{\varepsilon \to 0} \int_X \left(-\frac{\hat{\mathcal{L}} u}{u + \varepsilon} \right) f^2 \, dm \\ &\geq \inf_{x \in X} \left(-\frac{\hat{\mathcal{L}} u}{u} \right)(x) \int_X f^2 \, dm \\ &= \inf_{x \in X} \left(-\frac{\hat{\mathcal{L}} u}{u} \right)(x), \end{aligned}$$

which implies (5.10). □

M. F. Chen [13, Theorem 1.1] obtained the same estimate as Theorem 5.8 for jump processes on measurable spaces. For the case where the state spaces are locally compact, Theorem 5.8 becomes an extension of Chen's result to general symmetric Markov processes.

We shall give another lower bound estimate of λ. Define $G^0 u := u$ and

$$G^{n+1} u(x) = G \left(G^n u \right)(x) = E_x \left[\int_0^\infty G^n u(X_s) \, ds \right], \quad x \in X$$

for any nonnegative integer $n \geq 0$ and $u \in C_b^{\varphi,+}(X)$.

Proposition 5.9. *For any $n \geq 0$ and $u \in C_b^{\varphi,+}(X)$, it holds that*

$$\inf_{x \in X} \left(\frac{G^n u}{G^{n+1} u} \right)(x) \leq \inf_{x \in X} \left(\frac{G^{n+1} u}{G^{n+2} u} \right)(x). \tag{5.11}$$

Proof. Since

$$\begin{aligned} G^{n+1} u &= G \left(\frac{G^n u}{G^{n+1} u} G^{n+1} u \right) \\ &\geq \left(\inf_{x \in X} \left(\frac{G^n u}{G^{n+1} u} \right)(x) \right) G^{n+2} u, \end{aligned}$$

we get (5.11). □

Theorem 5.10. *It holds that*

$$\lambda_0 \geq \inf_{x \in X} \left(\frac{G^n u}{G^{n+1} u} \right)(x) \tag{5.12}$$

for any $n \geq 0$ and $u \in C_b^{\varphi,+}(X)$. In particular, it follows that, by taking $u = 1$ and $n = 0$,

$$\lambda_0 \geq \frac{1}{\sup_{x \in X} E_x[\zeta]}. \tag{5.13}$$

Proof. Since, by the definition of $\hat{\mathcal{L}}$,

$$-\hat{\mathcal{L}}G^{n+1}u = G^n u$$

for any $u \in C_b^{\varphi,+}(X)$, it is clear that $G^{n+1}u \in \mathcal{C}$. Applying Theorem 5.8 to $G^{n+1}u$, we obtain (5.12). $\square$

Using Proposition 5.9 and Theorem 5.10, we have

Corollary 5.11. *It holds that*

$$\lambda_0 \geq \lim_{n\to\infty} \inf_{x\in X} \left(\frac{G^n u}{G^{n+1}u}\right)(x)$$

for any $u \in C_b^{\varphi,+}(X)$ $\square$

Under some assumptions, S. Sato [48] gave the same estimate of the spectral radius for non-symmetric right continuous strong Markov processes by using the dual operator of the resolvent.

From now on, we suppose in addition that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular. Using Theorem 5.4, we shall prove the following:

Theorem 5.12. ([7], [28], [50], [59]) *For any* $\mu \in \mathcal{S}_1$, *it holds that*

$$\int_X f^2\, d\mu \leq \|G\mu\|_\infty \mathcal{E}(f,f), \quad f \in \mathcal{F}. \tag{5.14}$$

There are analytic and probabilistic approaches to prove (5.14): Vondraček [59, Theorem 1] derived (5.14) from the capacitary inequality; however, the constant of the right hand side is $4\|G\mu\|_\infty$ instead of $\|G\mu\|_\infty$. Stollmann and Voigt [50, Theorem 3.1] first proved (5.14) by using the operator theory. Fitzsimmons [28, Example 1.17] also established (5.14) from Hardy's inequality for Dirichlet forms ([28, Theorem 1.9]). In [7, Corollary 3.1], Ben Amor showed (5.14) by using the fact that the measure $|u|\cdot\mu$ is of finite energy integral for $u \in L^2(X;\mu)$ ([7, Theorem 3.1]). Here we give a new proof of (5.14) by applying Theorem 5.4 to the time changed process $\check{\mathbf{M}}$ of $\mathbf{M}$ with respect to the PCAF A_t^μ. Recall that $\left(\check{\mathcal{E}}, \check{\mathcal{F}}\right)$ is the Dirichlet form of $\check{\mathbf{M}}$.

Proof. Since $\mathcal{E}(f,f) \geq \mathcal{E}(|f|,|f|)$, it suffices to prove (5.14) for $f \in \mathcal{F}$ with $f \geq 0$ μ-a.e. on X. The Dirichlet principle (1.9) implies that

$$\mathcal{E}(f,f) \geq \check{\mathcal{E}}(f|_F, f|_F),$$

where $f|_F$ is the restriction of f on $F = \mathrm{supp}[\mu]$. Let $\check{\mathcal{L}}$ be the extended generator of $\check{\mathbf{M}}$. Then it follows from Theorem 5.4 that

$$\check{\mathcal{E}}(f|_F, f|_F) = -\inf_{u\in\mathcal{D}^+(\check{\mathcal{L}}),\ \varepsilon>0} \int_F \frac{\check{\mathcal{L}}u}{u+\varepsilon}\,(f|_F)^2 d\mu.$$

Let $\check{G}$ be the 0-resolvent of $\check{\mathbf{M}}$. Since $\check{G}\mathbf{1}(x) = E_x\left[A_\zeta^\mu\right]$, (1.5) yields that

$$\begin{aligned}\check{\mathcal{E}}(f|_F, f|_F) &\geq \int_F \frac{1}{\check{G}\mathbf{1}+\varepsilon}\,(f|_F)^2 d\mu \\ &\geq \frac{1}{\|G\mu\|_\infty+\varepsilon}\int_F f^2 d\mu.\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we have (5.14). $\square$

From now on, we assume in addition that the transition density of $\mathbf{M}$ is absolutely continuous with respect to the measure m. Let $\mu \in \mathcal{K}_\infty$. Then it follows from (5.13) that, if

$$\sup_{x \in X} E_x \left[A_\zeta^\mu \right] < 1, \tag{5.15}$$

then $\check{\lambda}(\mu) > 1$, where $\check{\lambda}(\mu)$ is the bottom of the spectrum for $\check{\mathbf{M}}$ as defined in (1.11). We thus rediscover the Khas'minskii lemma [38, Lemma 3] by Theorem 1.2: the condition (5.15) implies that

$$\sup_{x \in X} E_x \left[\exp\left(A_\zeta^\mu \right) \right] < \infty.$$

Chapter 6

Principal eigenvalues for symmetric α-stable processes

In this chapter, we estimate the principal eigenvalues for symmetric α-stable processes by using generalized Barta's inequality. Furthermore, we calculate explicitly the principal eigenvalues for time changed processes of Brownian motions and symmetric α-stable processes, and of Schrödinger operators.

6.1 Principal eigenvalues for time changed processes

6.1.1 In case of $\alpha = 2$

We first calculate the principal eigenvalues for time changed processes of killed Brownian motions. In this subsection, we denote by $\mathbf{M} = (B_t, P_x)$ the Brownian motion on $\mathbb{R}^d$. For a measure $\nu \in \mathcal{K}^{\mathbb{R}^d}$, let $\mathbf{M}^\nu = (B_t^\nu, P_x^\nu)$ be the $\exp(-A^\nu)$-subprocess of the Brownian motion on $\mathbb{R}^d$ and $G^\nu(x, y)$ the Green function of $\mathbf{M}^\nu$. Define for a measure $\mu \in \mathcal{K}_\infty^{\mathbb{R}^d}$,

$$\check{\lambda}(\mu, \nu) = \inf\left\{\frac{1}{2}\mathbf{D}(u, u) + \int_{\mathbb{R}^d} u^2\, d\nu : u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2\, d\mu = 1\right\}.$$

Then the equation (1.9) implies that $\check{\lambda}(\mu, \nu)$ coincides with the principal eigenvalue for the time changed process of $\mathbf{M}^\nu$ with respect to A^μ.

For $d = 1$, the Dirac measure δ_a at $a \in \mathbb{R}$ admits the local time $l_a(t)$ at a under the Revuz correspondence ([29, Examples 2.1.2 and 5.1.1]). For $d \geq 2$, since the space with codimension one is of positive capacity, the surface measure also admits the local time on the surface.

Example 6.1. Assume that $d = 1$. If we set $\nu(dx) = \mathbf{1}_{(a,b)}\, dx$ for $a < b$, then $A_t^{\alpha\nu} = \alpha \int_0^t \mathbf{1}_{(a,b)}(B_s)\, ds$ for $\alpha > 0$. By definition,

$$\check{\lambda}(\beta\delta_z, \alpha\mathbf{1}_{(a,b)}dx) = \inf\left\{\frac{1}{2}\mathbf{D}(u, u) + \alpha\int_a^b u^2\, dx : u \in C_0^\infty(\mathbb{R}),\ \beta u^2(z) = 1\right\}. \tag{6.1}$$

Let Cap be the 0-order capacity with respect to $\mathbf{M}^{\alpha\nu}$. Then the infimum above is attained by

$$\frac{1}{\sqrt{\beta}} P_x^{\alpha\nu}(\sigma_z < \infty) = \frac{1}{\sqrt{\beta}} E_x\left[\exp\left(-\alpha\int_0^{\sigma_z} \mathbf{1}_{(a,b)}(B_s)\, ds\right)\right]$$

because the right hand side of (6.1) coincides with $\mathrm{Cap}(\{z\})/\beta$. First suppose that $z < a$. Then it follows from [11, p.167, 2.7.1] that

$$E_x\left[\exp\left(-\alpha\int_0^{\sigma_z}\mathbf{1}_{(a,b)}(B_s)\,ds\right)\right]$$
$$=\begin{cases} 1, & x<z \\ \dfrac{\sqrt{2\alpha}(a-x)\sinh(\sqrt{2\alpha}(b-a))+\cosh(\sqrt{2\alpha}(b-a))}{\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a))+\cosh(\sqrt{2\alpha}(b-a))}, & z<x\le a \\ \dfrac{\cosh(\sqrt{2\alpha}(b-x))}{\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a)+\cosh(\sqrt{2\alpha}(b-a))}, & a\le x\le b \\ \dfrac{1}{\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a))+\cosh(\sqrt{2\alpha}(b-a))}, & b\le x. \end{cases}$$

Hence a direct calculation yields that

$$\check{\lambda}(\beta\delta_z,\alpha\mathbf{1}_{(a,b)}\,dx)=\frac{1}{2\beta}\frac{\sqrt{2\alpha}\sinh(\sqrt{2\alpha}(b-a))}{\cosh(\sqrt{2\alpha}(b-a))+\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a))}.$$

Next suppose that $a < z \le b$. It also follows from [11, p.167, 2.7.1] that

$$E_x\left[\exp\left(-\alpha\int_0^{\sigma_z}\mathbf{1}_{(a,b)}(B_s)\,ds\right)\right]=\begin{cases} \dfrac{1}{\cosh(\sqrt{2\alpha}(z-a))}, & x\le a \\ \dfrac{\cosh(\sqrt{2\alpha}(x-a))}{\cosh(\sqrt{2\alpha}(z-a))}, & a\le x<z \\ \dfrac{\cosh(\sqrt{2\alpha}(b-x))}{\cosh(\sqrt{2\alpha}(b-z))}, & z<x\le b \\ \dfrac{1}{\cosh(\sqrt{2\alpha}(b-z))}, & b\le x. \end{cases}$$

Thereby,

$$\check{\lambda}(\beta\delta_z,\alpha\mathbf{1}_{(a,b)}dx)=\frac{\sqrt{\alpha}}{4\sqrt{2}\beta}\left\{\frac{\sinh(2\sqrt{2\alpha}(z-a))}{\cosh^2(\sqrt{2\alpha}(z-a))}+\frac{\sinh(2\sqrt{2\alpha}(b-z))}{\cosh^2(\sqrt{2\alpha}(b-z))}\right\}.$$

Example 6.2. First suppose that $d = 1$. For $n \in \mathbb{N}$, let $\{a_i\}_{i=0}^n$ and $\{b_i\}_{i=1}^n$ be sequences which satisfy $a_0 < b_1 < a_1 < b_2 < \cdots < b_n < a_n$. If we set $\nu = \sum_{i=0}^n \alpha_i\delta_{a_i}$ for $\alpha_i \ge 0$, then $A_t^\nu = \sum_{i=0}^n \alpha_i l_{a_i}(t)$. Put $\mu = \sum_{i=1}^n \beta_i\delta_{b_i}$ for $\beta_i > 0$. Then

$$\check{\lambda}\left(\sum_{i=1}^n\beta_i\delta_{b_i},\sum_{i=0}^n\alpha_i\delta_{a_i}\right)=\inf\left\{\mathcal{E}(u,u)+\sum_{i=0}^n\alpha_iu(a_i)^2 : u\in C_0^\infty(\mathbb{R}),\ \sum_{i=1}^n\beta_iu(b_i)^2=1\right\}.$$

Note that the infimum above is attained by the harmonic function u, which satisfies

$$u(x)=E_x\left[\exp\left(-A^\nu_{\sigma_{\overline{B}}}\right)u(B_{\sigma_{\overline{B}}})\right]$$
$$=\begin{cases} u(b_1)E_x\left[\exp\left(-\alpha_0 l_{a_0}(\sigma_1)\right)\right], & x<b_1 \\ u(b_i)E_x\left[\exp\left(-\alpha_i l_{a_i}(\sigma_i)\right):\sigma_i<\sigma_{i+1}\right] & \\ \qquad +u(b_{i+1})E_x\left[\exp\left(-\alpha_i l_{a_i}(\sigma_{i+1})\right):\sigma_{i+1}<\sigma_i\right], & b_i<x<b_{i+1} \\ u(b_n)E_x\left[\exp\left(-\alpha_n l_{a_n}(\sigma_n)\right)\right], & b_n<x. \end{cases}$$

Here $\overline{B} = \{b_i\}_{i=1}^n$ and σ_i is the hitting time of b_i. Then it follows from [11, p.164, 2.3.1] that

$$E_x\left[\exp\left(-\alpha_0 l_{a_0}(\sigma_1)\right)\right] = \frac{1+2\alpha_0(x-a_0)}{1+2\alpha_0(b_1-a_0)},\ a_0 \le x < b_1$$
$$E_x\left[\exp\left(-\alpha_n l_{a_n}(\sigma_n)\right)\right] = \frac{1+2\alpha_n(a_n-x)}{1+2\alpha_n(a_n-b_n)},\ b_n < x \le a_n.$$

It also follows from [11, p.174, 3.3.5] that

$$E_x\left[\exp\left(-\alpha_i l_{a_i}(\sigma_i)\right) : \sigma_i < \sigma_{i+1}\right] = \begin{cases} \dfrac{b_{i+1}-x+2\alpha_i(b_{i+1}-a_i)(a_i-x)}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}, & b_i < x \le a_i \\ \dfrac{b_{i+1}-x}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}, & a_i \le x < b_{i+1}, \end{cases}$$

and

$$E_x\left[\exp\left(-\alpha_i l_{a_i}(\sigma_{i+1})\right) : \sigma_{i+1} < \sigma_i\right] = \begin{cases} \dfrac{x-b_i}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}, & b_i < x \le a_i \\ \dfrac{x-b_i+2\alpha_i(a_i-b_i)(x-a_i)}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}, & a_i \le x < b_{i+1}. \end{cases}$$

We thus have

$$\begin{aligned}
&\frac{1}{2}\mathbf{D}(u,u) + \sum_{i=1}^n \alpha_i u(a_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{n-1} \frac{1}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}(u(b_{i+1})-u(b_i))^2 \\
&+ \left(\frac{\alpha_0}{1+2\alpha_0(b_1-a_0)} + \frac{\alpha_1(b_2-a_1)}{b_2-b_1+2\alpha_1(b_2-a_1)(a_1-b_1)}\right)u(b_1)^2 \\
&+ \sum_{i=2}^{n-1}\left(\frac{\alpha_{i-1}(a_{i-1}-b_{i-1})}{b_i-b_{i-1}+2\alpha_i(b_i-a_{i-1})(a_{i-1}-b_{i-1})} + \frac{\alpha_i(b_{i+1}-a_i)}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}\right)u(b_i)^2 \\
&+ \left(\frac{\alpha_{n-1}(a_{n-1}-b_{n-1})}{b_n-b_{n-1}+2\alpha_{n-1}(b_n-a_{n-1})(a_{n-1}-b_{n-1})} + \frac{\alpha_n}{1+2\alpha_n(a_n-b_n)}\right)u(b_n)^2.
\end{aligned} \tag{6.2}$$

Here we note that the right hand side of (6.2) is the Dirichlet form on $L^2(\overline{B};\mu)$ generated by the time changed process of $\mathbf{M}^\nu$ with respect to A_t^μ. Moreover, its Q-matrix is

$$Q = \begin{pmatrix} {\beta_1}^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & {\beta_2}^{-1} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & {\beta_{n-1}}^{-1} & 0 \\ 0 & \cdots & \cdots & 0 & {\beta_n}^{-1} \end{pmatrix} \begin{pmatrix} -B_1 & A_1 & 0 & \cdots & \cdots & 0 \\ A_1 & -B_2 & A_2 & \cdots & \cdots & \cdots \\ 0 & A_2 & -B_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & A_{n-2} & -B_{n-1} & A_{n-1} \\ 0 & \cdots & \cdots & 0 & A_{n-1} & -B_n \end{pmatrix},$$

where

$$
\begin{aligned}
A_k &= \frac{1}{2\left(b_{k+1} - b_k + 2\alpha_i(b_{k+1} - a_k)(a_k - b_k)\right)} \\
B_1 &= \frac{\alpha_0}{1 + 2\alpha_0(b_1 - a_0)} + A_1(1 + 2\alpha_1(b_2 - a_1)) \\
B_k &= A_{k-1}(1 + 2\alpha_{k-1}(a_{k-1} - b_{k-1})) + A_k(1 + 2\alpha_k(b_{k+1} - a_k)), \ 2 \le k \le n-1 \\
B_n &= \frac{\alpha_n}{1 + 2\alpha_n(a_n - b_n)} + A_{n-1}(1 + 2\alpha_{n-1}(a_{n-1} - b_{n-1})).
\end{aligned}
$$

Hence

$$
\check{\lambda}\left(\sum_{i=1}^{n} \beta_i \delta_{b_i}, \sum_{i=0}^{n} \alpha_i \delta_{a_i}\right) = -\max\{\kappa : |Q - \kappa I| = 0\},
$$

where I is the $n \times n$-unit matrix. When $n = 1$, we get

$$
\check{\lambda}\left(\beta_1 \delta_{b_1}, \alpha_0 \delta_{a_0} + \alpha_1 \delta_{a_1}\right) = \frac{\alpha_0 + \alpha_1 + 2\alpha_0\alpha_1(a_1 - a_0)}{\beta_1(1 + 2\alpha_0(b_1 - a_0))(1 + 2\alpha_1(a_1 - b_1))}.
$$

In particular, if $b_1 - a_0 = a_1 - b_1 = r$, then

$$
\check{\lambda}\left(\beta_1 \delta_{b_1}, \alpha_0 \delta_{a_0} + \alpha_1 \delta_{a_1}\right) = \frac{\alpha_0 + \alpha_1 + 4\alpha_0\alpha_1 r}{\beta_1(1 + 2\alpha_0 r)(1 + 2\alpha_1 r)}.
$$

When $n = 2$ and $\alpha_0 = \alpha_2 = 0$, we obtain

$$
\begin{aligned}
\check{\lambda}\left(\beta_1 \delta_{b_1} + \beta_2 \delta_{b_2}, \alpha_1 \delta_{a_1}\right) =& \frac{\beta_1(1 + 2\alpha_1(a_1 - b_1)) + \beta_2(1 + 2\alpha_1(b_2 - a_1))}{4\beta_1\beta_2\{b_2 - b_1 + 2\alpha_1(b_2 - a_1)(a_1 - b_1)\}} \\
& - \frac{\sqrt{\{\beta_1(1 + 2\alpha_1(a_1 - b_1)) - \beta_2(1 + 2\alpha_1(b_2 - a_1))\}^2 + 4\beta_1\beta_2}}{4\beta_1\beta_2\{b_2 - b_1 + 2\alpha_1(b_2 - a_1)(a_1 - b_1)\}}.
\end{aligned}
$$

Assume in addition that $\beta_1 = \beta_2 = \beta$ and $b_2 - a_1 = a_1 - b_1 = r$. Then

$$
\check{\lambda}\left(\beta(\delta_{b_1} + \delta_{b_2}), \alpha_1 \delta_{a_1}\right) = \frac{\alpha_1}{2\beta(1 + \alpha_1 r)}.
$$

Next suppose that $d \ge 2$. Let $\{r_i\}_{i=0}^n$ and $\{R_i\}_{i=1}^n$ be sequences such that $0 < r_0 < R_1 < r_1 < R_2 < \cdots < R_n < r_n$. Denote by δ_r the surface measure on $\partial B(r) = \{x \in \mathbb{R}^d : |x| = r\}$. We now calculate the following:

$$
\begin{aligned}
&\check{\lambda}\left(\sum_{i=1}^{n} \beta_i \delta_{R_i}, \sum_{i=0}^{n} \alpha_i \delta_{r_i}\right) \\
&= \inf\left\{\frac{1}{2}\mathbf{D}(u,u) + \sum_{i=0}^{n} \alpha_i \int_{\partial B(r_i)} u^2 \, d\delta_{r_i} : u \in C_0^\infty(\mathbb{R}^d), \ \sum_{i=1}^{n} \beta_i \int_{\partial B(R_i)} u^2 \, d\delta_{R_i} = 1\right\}.
\end{aligned}
$$

Because of the spherical symmetry, it suffices for us to consider the Bessel process, $W_t = |B_t|$. Then the right hand side above is equal to

$$
\inf\left\{\frac{1}{2}\int_0^\infty \left(\frac{du}{dx}\right)^2 x^{d-1} dx + \sum_{i=0}^{n} \alpha_i u(r_i)^2 r_i^{d-1} : u \in C_0^\infty([0,\infty)), \ \sum_{i=1}^{n} \beta_i u(R_i)^2 R_i^{d-1} = 1\right\}.
$$

Hence we can calculate $\check{\lambda}\left(\sum_{i=1}^n \beta_i\delta_{R_i}, \sum_{i=0}^n \alpha_i\delta_{r_i}\right)$ by the same way as for the one-dimensional case. For example, when $d = 2$ and $n = 1$,

$$\check{\lambda}\left(\beta_1\delta_{R_1}, \alpha_0\delta_{r_0} + \alpha_1\delta_{r_1}\right) = \frac{\alpha_0 r_0 + \alpha_1 r_1 + 2\alpha_0\alpha_1 r_0 r_1(\log r_1 - \log r_0)}{\beta_1 R_1 \left\{1 + 2\alpha_0 r_0(\log R_1 - \log r_0)\right\}\left\{1 + 2\alpha_1 r_1(\log r_1 - \log R_1)\right\}}.$$

On the other hand, when $d \geq 3$ and $n = 1$,

$$\check{\lambda}\left(\beta_1\delta_{R_1}, \alpha_0\delta_{r_0} + \alpha_1\delta_{r_1}\right) = \frac{1}{\beta_1 R_1^{2\nu+1}}\left\{\frac{\nu\alpha_0 r_0^{2\nu+1}}{\nu + \alpha_0 r_0^{2\nu+1}(r_0^{-2\nu} - R_1^{-2\nu})} + \frac{2\nu(\nu + \alpha_1 r_1)R_1^{2\nu}}{\nu + \alpha_1 r_1 R_1^{2\nu}(R_1^{-2\nu} - r_1^{-2\nu})}\right\},$$

where $\nu = d/2 - 1$.

6.1.2 In case of $0 < \alpha \leq 2$

We next consider the principal eigenvalues for time changed processes of symmetric α-stable processes. Let $\mathbf{M}^\alpha = (X_t, P_x)$ be the symmetric α-stable process on $\mathbb{R}^d$ and $\mathbf{M}^D$ the absorbing α-stable process on an open set D in $\mathbb{R}^d$. Take $\mu \in \mathcal{K}_\infty^D$ and let $\check{\mathbf{M}}^D$ be the time changed process of $\mathbf{M}^D$ with respect to A_t^μ. Let

$$\check{\lambda}(\mu; D) = \inf\left\{\mathcal{E}^D(u,u) : u \in C_0^\infty(D), \int_D u^2\, d\mu = 1\right\}.$$

Then $\check{\lambda}(\mu; D)$ is the principal eigenvalue for $\check{\mathbf{M}}^D$ as mentioned in Chapter 1.

Example 6.3. Let $B(R) = \{x \in \mathbb{R}^d : |x| < R\}$. Denote by τ_R the exit time from $B(R)$, $\tau_R = \inf\{t > 0 : X_t \notin B(R)\}$. Let $\mathbf{M}^R$ be the symmetric α-stable process killed outside $B(R)$. Since

$$E_x\left[\tau_R\right] = \frac{2^{1-\alpha}\Gamma\left(\dfrac{d}{2}\right)}{\Gamma\left(\dfrac{\alpha}{2}+1\right)\Gamma\left(\dfrac{d+\alpha}{2}\right)}\left(R^2 - |x|^2\right)^{\alpha/2}, \quad x \in B(R)$$

by Section 5 of [30], we have by (5.13),

$$\check{\lambda}(dx; B(R)) \geq \frac{\Gamma\left(\dfrac{\alpha}{2}+1\right)\Gamma\left(\dfrac{d+\alpha}{2}\right)}{2^{1-\alpha}\Gamma\left(\dfrac{d}{2}\right)R^\alpha}.$$

In [6, §3], the same estimate above was obtained in a similar fashion.

Example 6.4. For $d > \alpha$, let $\mu(dx) = \mathbf{1}_{B(R)}dx$ be the Lebesgue measure restricted on $B(R)$. Then the PCAF A_t^μ with Revuz measure μ is given by

$$A_t^\mu = \int_0^t \mathbf{1}_{B(R)}(X_s)\, ds.$$

Then A_∞^μ is the lifetime of the time changed process of $\mathbf{M}^\alpha$ with respect to A^μ. Let ω_d be the surface area of a unit ball in $\mathbb{R}^d$. A direct calculation yields that

$$\begin{aligned}\sup_{x\in B(R)} E_x[A_\infty^\mu] &= \sup_{x\in B(R)}\int_{B(R)} G(x,y)\, dy \\ &= \int_{B(R)} G(0,y)\, dy = \frac{2^{1-\alpha}\Gamma\left(\dfrac{d-\alpha}{2}\right)\omega_d}{\alpha\pi^{d/2}\Gamma\left(\dfrac{\alpha}{2}\right)}R^\alpha.\end{aligned}$$

Here $G(x,y)$ is the Green function of $\mathbf{M}^\alpha$ in (1.22). Noting that $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$, we obtain

$$\check{\lambda}(\mathbf{1}_{B(R)}dx;\mathbb{R}^d) \geq \frac{\alpha\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)}{2^{1-\alpha}\Gamma\left(\frac{d-\alpha}{2}\right)R^\alpha}.$$

In the reminder of this subsection, we assume that $d=1$ and $1<\alpha\leq 2$. Then, since one point is of positive capacity, the Dirac measure δ_a at $a\in\mathbb{R}$ admits the local time at a under the Revuz correspondence.

Example 6.5. Let $\mathbf{M}^R$ be the absorbing symmetric α-stable process on $(-R,R)$ and $a\in(-R,R)$. Denote by $G^R(x,y)$ the Green function of $\mathbf{M}^R$. Since

$$\begin{aligned}G^R(x,a) &= E_x\left[l_a(\tau_R)\right] = E_x\left[l_a(\tau_R);\sigma_a<\tau_R\right]\\&= E_x\left[E_{X_{\sigma_a}}\left[l_a(\tau_R)\right];\sigma_a<\tau_R\right]\\&= P_x(\sigma_a<\tau_R)G^R(a,a),\end{aligned}$$

that is,

$$P_x(\sigma_a<\tau_R) = \frac{G^R(x,a)}{G^R(a,a)},$$

we see in a similar way to Example 6.1 that

$$\begin{aligned}\check{\lambda}(\delta_a;(-R,R)) &= \mathcal{E}^\alpha(P_\cdot(\sigma_a<\tau_R),P_\cdot(\sigma_a<\tau_R))\\&= \frac{1}{G^R(a,a)^2}\mathcal{E}^\alpha(G^R(\cdot,a),G^R(\cdot,a))\\&= \frac{1}{G^R(a,a)^2}\int_{-R}^R G^R(x,a)\,\delta_a(dx) = \frac{1}{G^R(a,a)}.\end{aligned}$$

It follows from Corollary 4 of [10] that, for $|x|<R$ and $|y|\leq R$,

$$G^R(x,y) = \frac{1}{2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)^2}\int_0^z (u+1)^{-1/2}u^{\alpha/2-1}\,du|x-y|^{\alpha-1},$$

where $z=(R^2-|x|^2)(R^2-|y|^2)/R^2|x-y|^2$. Hence

$$G^R(a,a) = \frac{(R^2-a^2)^{\alpha-1}}{(\alpha-1)2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2R^{\alpha-1}},$$

and

$$\check{\lambda}(\delta_a;(-R,R)) = \frac{(\alpha-1)2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2R^{\alpha-1}}{(R^2-a^2)^{\alpha-1}}.$$

Let $\mathbf{M}^\infty$ be the absorbing symmetric α-stable process on $(0,\infty)$ and $a\in(0,\infty)$. Denote by $G^\infty(x,y)$ the Green function of $\mathbf{M}^\infty$. Since

$$G^\infty(x,y) = \frac{2}{\Gamma\left(\frac{\alpha}{2}\right)^2}\int_0^{x\wedge y} z^{(\alpha-2)/2}\left(z+|y-x|\right)^{(\alpha-2)/2}\,dz$$

by [45], we have

$$\check{\lambda}(\delta_a; (0,\infty)) = \frac{(\alpha-1)\Gamma\left(\frac{\alpha}{2}\right)^2}{2a^{\alpha-1}}.$$

Example 6.6. Let $\mathbf{M}^0$ be the absorbing α-stable process on $\mathbb{R}\setminus\{0\}$ and denote by G^0 the Green function of $\mathbf{M}^0$. Getoor [31] then showed that

$$G^0(x,y) = -\frac{1}{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}\left(|x|^{\alpha-1}+|y|^{\alpha-1}-|x-y|^{\alpha-1}\right)$$

(see also [44, p. 379]). Hence for $a>0$,

$$\begin{aligned}\check{\lambda}(\delta_a;\mathbb{R}\setminus\{0\}) &= \frac{1}{G^0(a,a)}\\ &= -\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{2a^{\alpha-1}}.\end{aligned}$$

The following are three graphs of $\check{\lambda}(\delta_a;\mathbb{R}\setminus\{0\})$ with respect to $\alpha\in(1,2]$. If a is small, then $\check{\lambda}(\delta_a;\mathbb{R}\setminus\{0\})$ is increasing monotonously. However, $\check{\lambda}(\delta_a;\mathbb{R}\setminus\{0\})$ takes the maximal value for large a. We can guess that $\check{\lambda}(\delta_a;\mathbb{R}\setminus\{0\})$ takes the maximal value for $a>1.5$.

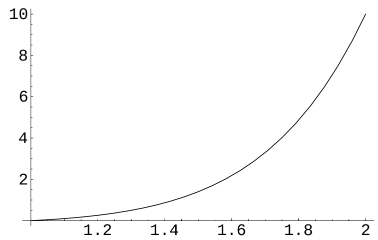

Figure 6.1: $\check{\lambda}(\delta_{0.05};\mathbb{R}\setminus\{0\})$

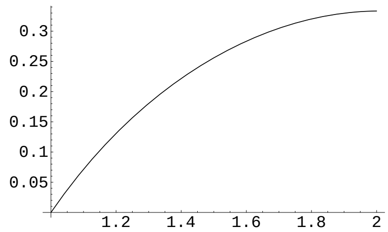

Figure 6.2: $\check{\lambda}(\delta_{1.5};\mathbb{R}\setminus\{0\})$

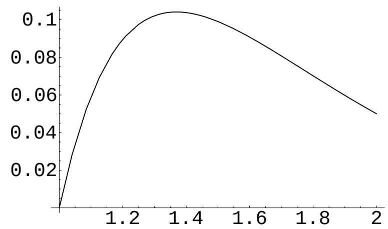

Figure 6.3: $\check{\lambda}(\delta_{10};\mathbb{R}\setminus\{0\})$

We can also calculate $\check{\lambda}(\delta_a+\delta_{-a};\mathbb{R}\setminus\{0\})$. In fact, the strong Markov property implies that

$$\begin{aligned}G^0(x,a)+G^0(x,-a) &= E_x\left[(l_a(\sigma_0)+l_{-a}(\sigma_0))\right]\\ &= E_x\left[\left(E_{X_{\sigma_a\wedge\sigma_{-a}}}\left[l_a(\sigma_0)\right]+E_{X_{\sigma_a\wedge\sigma_{-a}}}\left[l_{-a}(\sigma_0)\right]\right);\sigma_a\wedge\sigma_{-a}<\sigma_0\right]\\ &= E_x\left[\left(E_a\left[l_a(\sigma_0)\right]+E_a\left[l_{-a}(\sigma_0)\right]\right);\sigma_a\wedge\sigma_{-a}<\sigma_0,\sigma_a<\sigma_{-a}\right]\\ &\quad + E_x\left[\left(E_{-a}\left[l_a(\sigma_0)\right]+E_{-a}\left[l_{-a}(\sigma_0)\right]\right);\sigma_a\wedge\sigma_{-a}<\sigma_0,\sigma_{-a}<\sigma_a\right].\end{aligned}$$

By noting that $G^0(a,a)=G^0(-a,-a)$ and $G^0(a,-a)=G^0(-a,a)$, the right hand side above is equal to

$$\begin{aligned}&G^0(a,a)\left(P_x(\sigma_a\wedge\sigma_{-a}<\sigma_0,\sigma_a<\sigma_{-a})+P_x(\sigma_a\wedge\sigma_{-a}<\sigma_0,\sigma_{-a}<\sigma_a)\right)\\ &+G^0(a,-a)\left(P_x(\sigma_a\wedge\sigma_{-a}<\sigma_0,\sigma_a<\sigma_{-a})+P_x(\sigma_a\wedge\sigma_{-a}<\sigma_0,\sigma_{-a}<\sigma_a)\right)\\ &=\left(G^0(a,a)+G^0(a,-a)\right)P_x(\sigma_a\wedge\sigma_{-a}<\sigma_0),\end{aligned}$$

that is,

$$P_x(\sigma_a\wedge\sigma_{-a}<\sigma_0) = \frac{G^0(x,a)+G^0(x,-a)}{G^0(a,a)+G^0(a,-a)}. \tag{6.3}$$

A direct calculation implies that

$$\begin{aligned}\check{\lambda}(\delta_a+\delta_{-a};\mathbb{R}\setminus\{0\}) &= \inf\left\{\mathcal{E}^\alpha(u,u) : u\in C_0^\infty(\mathbb{R}\setminus\{0\}), u(a)^2+u(-a)^2=1\right\} \\ &= \inf\left\{\mathcal{E}^\alpha(u,u) : u\in C_0^\infty(\mathbb{R}\setminus\{0\}), u(a)=u(-a)=\frac{1}{\sqrt{2}}\right\} \\ &= \frac{1}{2}\inf\left\{\mathcal{E}^\alpha(u,u) : u\in C_0^\infty(\mathbb{R}\setminus\{0\}), u(a)=u(-a)=1\right\} \\ &= \frac{1}{2}\mathcal{E}^\alpha(P_\cdot(\sigma_a\wedge\sigma_{-a}<\sigma_0), P_\cdot(\sigma_a\wedge\sigma_{-a}<\sigma_0)).\end{aligned}$$

By (6.3), the last term above is equal to

$$\begin{aligned}&\frac{1}{2\left(G^0(a,a)+G^0(a,-a)\right)^2}\mathcal{E}^\alpha\left(G^0(\cdot,a)+G^0(\cdot,-a), G^0(\cdot,a)+G^0(\cdot,-a)\right) \\ &= \frac{1}{2\left(G^0(a,a)+G^0(a,-a)\right)^2}\int_{\mathbb{R}\setminus\{0\}}\left(G^0(x,a)+G^0(x,-a)\right)\left(\delta_a(dx)+\delta_{-a}(dx)\right) \\ &= \frac{1}{G^0(a,a)+G^0(a,-a)} = -\frac{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}{(4-2^{\alpha-1})a^{\alpha-1}}.\end{aligned}$$

Example 6.7. Let $\mathbf{M}^p$ be the absorbing symmetric α-stable process on $\mathbb{R}\setminus\{-p,p\}$. Denote by $G^p(x,y)$ the Green function of $\mathbf{M}^p$. We then see from (2.9) of [44] that

$$G^p(x,y) = L_p(x)+P_x(\sigma_p<\sigma_{-p})a(y-p)+P_x(\sigma_{-p}<\sigma_p)a(y+p)-a(y-x),$$

where

$$a(x) = -\frac{1}{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}|x|^{\alpha-1}$$

and L_p is some function. Noting that $G^p(x,p)=G^p(x,-p)=0$, we obtain

$$L^p(x) = \frac{1}{2}\left(a(x-p)+a(x+p)-a(2p)\right).$$

Since Theorem 6.5 of [31] yields that

$$P_x(\sigma_{\pm p}<\sigma_{\mp p}) = \frac{1}{2}+\frac{1}{2a(2p)}(a(x\pm p)-a(x\mp p)),$$

we get

$$\begin{aligned}G^p(x,y) &= \frac{1}{2}(a(x-p)+a(x+p)+a(y-p)+a(y+p)-a(2p)) \\ &\quad -\frac{1}{2a(2p)}(a(x-p)-a(x+p))(a(y-p)-a(y+p))-a(x-y).\end{aligned}$$

Therefore,

$$\begin{aligned}\check{\lambda}(\delta_q;\mathbb{R}\setminus\{-p,p\}) &= \frac{1}{G^p(q,q)} \\ &= -\frac{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)|2p|^{\alpha-1}}{4|p-q|^{\alpha-1}|p+q|^{\alpha-1}-\left(|p-q|^{\alpha-1}+|p+q|^{\alpha-1}-|2p|^{\alpha-1}\right)^2}\end{aligned}$$

for $p \neq q$. In particular,

$$\check{\lambda}(\delta_0; \mathbb{R} \setminus \{-p, p\}) = -\frac{\Gamma(\alpha) \cos\left(\frac{\pi\alpha}{2}\right)}{(2 - 2^{\alpha-2})|p|^{\alpha-1}}.$$

We can also see that

$$\begin{aligned}\check{\lambda}(\delta_q + \delta_{-q}; \mathbb{R} \setminus \{-p, p\}) &= \frac{1}{G^p(q,q) + G^p(q,-q)} \\ &= -\frac{\Gamma(\alpha) \cos\left(\frac{\pi\alpha}{2}\right)}{2|p-q|^{\alpha-1} + 2|p+q|^{\alpha-1} - |2p|^{\alpha-1} - |2q|^{\alpha-1}}.\end{aligned}$$

6.2 Principal eigenvalues of Schrödinger operators

In this subsection, we calculate the principal eigenvalues of Schrödinger operators defined by

$$\lambda_1(\mu; D) = \inf\left\{\mathcal{E}^D(u,u) - \int_D u^2\, d\mu : u \in C_0^\infty(D),\ \int_D u^2\, dx = 1\right\}$$

for $\mu \in \mathcal{K}_\infty^D$.

6.2.1 In case of $\alpha = 2$

Example 6.8. Suppose that $d = 1$. Let us take first $D = (-R, R)$ for $R > 0$ and $\mu = \sum_{i=1}^n \alpha_i \delta_{a_i}$, where $\alpha_i > 0$ and $-R < a_1 < a_2 < \cdots < a_n < R$. Denote by h the ground state corresponding to the principal eigenvalue $\lambda_1 := \lambda_1(\mu; (-R, R))$. We then see from (1.16) that

$$h(x) = \sum_{i=1}^n \alpha_i G_{-\lambda_1}^R(x, a_i) h(a_i),$$

where $G_\beta^R(x,y)$ is the β-resolvent of the absorbing Brownian motion on $(-R, R)$. Let G_β^R be the $n \times n$-matrix defined by $\left(\alpha_j G_\beta^R(a_i, a_j)\right)_{1 \le i,j \le n}$. Then the relation above implies that

$$\lambda_1 = \min\left\{\kappa : |G_{-\kappa}^R - I| = 0\right\}.$$

First take $\mu = \delta_a$. Since

$$G_{-\lambda_1}^R(x,y) = \frac{2}{\sqrt{-2\lambda_1}\sinh(\sqrt{-2\lambda_1}R)} \sinh\left\{\sqrt{-2\lambda_1}(R-x)\right\} \sinh\left\{\sqrt{-2\lambda_1}(R+y)\right\} \qquad (6.4)$$

for $-R < y \le x < R$ ([11, p.105]) and $G_{-\lambda_1}^R(a,a) = 1$, λ_1 is a solution to

$$\frac{\sqrt{-2\lambda}(e^{2\sqrt{-2\lambda}R} - e^{-2\sqrt{-2\lambda}R})}{e^{2\sqrt{-2\lambda}R} + e^{-2\sqrt{-2\lambda}R} - e^{2\sqrt{-2\lambda}a} - e^{-2\sqrt{-2\lambda}a}} = 1. \qquad (6.5)$$

Denote by h the ground state of λ_1 so that $\int_{-R}^R h^2\, dx = 1$. Then

$$h(x) = \begin{cases} C_1 \sinh\left\{2\sqrt{-2\lambda_1}(R-a)\right\} \sinh\left\{2\sqrt{-2\lambda_1}(R+x)\right\}, & -R < x \le a \\ C_1 \sinh\left\{2\sqrt{-2\lambda_1}(R+a)\right\} \sinh\left\{2\sqrt{-2\lambda_1}(R-x)\right\}, & a < x < R, \end{cases}$$

where

$$
\begin{aligned}
C_1 &= C_1(a, R, \lambda_1)\\
&= 2(-8\lambda_1)^{1/4}\Big\{ \sinh^2\left\{2\sqrt{-2\lambda_1}(R+a)\right\}\left(\sinh\left\{4\sqrt{-2\lambda_1}(R-a)\right\} - 4\sqrt{-2\lambda_1}(R-a)\right)\\
&\quad + \sinh^2\left\{2\sqrt{-2\lambda_1}(R-a)\right\}\left(\sinh\left\{4\sqrt{-2\lambda_1}(R+a)\right\} - 4\sqrt{-2\lambda_1}(R+a)\right)\Big\}^{-1/2}.
\end{aligned}
\tag{6.6}
$$

For instance, suppose that $a = 0$. Since the equation (6.5) becomes

$$
\frac{\sqrt{-2\lambda}(e^{2\sqrt{-2\lambda}R} + 1)}{e^{2\sqrt{-2\lambda}R} - 1} = 1,
$$

we can find that if $R > 1$, then λ_1 is a unique solution to the equation above and $-1/2 < \lambda_1 < 0$. Otherwise, $\lambda_1 = 0$.

Next take $\mu = \delta_a + \delta_{-a}$ for $a \in (0, R)$. Let h be the normalized ground state corresponding to the principal eigenvalue $\lambda_1 := \lambda_1(\delta_a + \delta_{-a}; (-R, R))$ so that $\int_{-R}^{R} h^2\, dx = 1$. Then it follows from (6.4) that

$$
\frac{\sqrt{-2\lambda_1}\sinh\left(2\sqrt{-2\lambda_1}R\right)}{2\sinh\left\{\sqrt{-2\lambda_1}(R-a)\right\}\left(\sinh\left\{\sqrt{-2\lambda_1}(R-a)\right\} + \sinh\left\{\sqrt{-2\lambda_1}(R+a)\right\}\right)} = 1 \tag{6.7}
$$

and

$$
h(x) = \begin{cases} C_2\sinh\left\{2\sqrt{-2\lambda_1}(R+a)\right\}\sinh\left\{2\sqrt{-2\lambda_1}(R-a)\right\}\sinh\left\{2\sqrt{-2\lambda_1}(R+x)\right\}, & -R < x \le -a\\ C_2\sinh\left\{2\sqrt{-2\lambda_1}(R-a)\right\}\sinh\left\{2\sqrt{-2\lambda_1}(R-x)\right\}\sinh\left\{2\sqrt{-2\lambda_1}(R+x)\right\}, & -a < x \le a\\ C_2\sinh\left\{2\sqrt{-2\lambda_1}(R-a)\right\}\sinh\left\{2\sqrt{-2\lambda_1}(R+a)\right\}\sinh\left\{2\sqrt{-2\lambda_1}(R-x)\right\}, & a < x < R, \end{cases}
$$

where $C_2 = C_2(a, R, \lambda_1)$ is the normalizing positive constant. Assume that $a = 1$. If $R > 3/2$, then the principal eigenvalue λ_1 is a negative unique solution to (6.7). Otherwise, $\lambda_1 = 0$

Example 6.9. Suppose that $d = 1$. Let us take first $D = (0, \infty)$ and $a \in (0, \infty)$. Denote by $G^0_{-\beta}(x, y)$ the β-resolvent of the absorbing Brownian motion on $(0, \infty)$:

$$
G^0_\beta(x, y) = \frac{2}{\sqrt{2\beta}} e^{-\sqrt{2\beta}x}\sinh\left(\sqrt{2\beta}y\right)
$$

for $0 < y < x$ ([11, p.107]). By the same way as in Example 6.8, it follows that the principal eigenvalue $\lambda_1 := \lambda_1(\delta_a; (0, \infty))$ is a unique solution to

$$
\frac{\sqrt{-2\lambda}e^{2\sqrt{-2\lambda}a}}{e^{2\sqrt{-2\lambda}a} - 1} = 1.
$$

A direct calculation implies that this equation has a negative unique solution $-1/2 < \lambda_1 < 0$ if $a > 1/2$. We denote by h the ground state corresponding to λ_1 with normalization $\int_0^\infty h^2\, dx = 1$. Then

$$
h(x) = \begin{cases} C_3 e^{-\sqrt{-2\lambda_1}a}\sinh(\sqrt{-2\lambda_1}x), & 0 < x \le a\\ C_3 e^{-\sqrt{-2\lambda_1}x}\sinh(\sqrt{-2\lambda_1}a), & a < x, \end{cases}
$$

where

$$C_3 = C_3(a, \lambda_1) = -\frac{4\lambda_1}{\left(e^{2\sqrt{-2\lambda_1}a} - (1 + 2\sqrt{-2\lambda_1}a)\right)^{1/2}}.$$

Next take $D = (0, \infty)$ and $\mu = \delta_a + \delta_b$ for $0 < a < b$. Put $\lambda_1 = \lambda_1(\delta_a + \delta_b; (0, \infty))$. We then see in a similar way to Example 6.8 that

$$G^0_{-\lambda_1}(a, b)^2 = (1 - G^0_{-\lambda_1}(a, a))(1 - G^0_{-\lambda_1}(b, b)).$$

Denote by h the ground state corresponding to λ_1 with normalization $\int_0^\infty h^2\, dx = 1$. Then

$$h(x) = \begin{cases} C_4 e^{-\sqrt{-2\lambda_1}a}\{e^{2\sqrt{-2\lambda_1}a} + e^{2\sqrt{-2\lambda_1}b}(\sqrt{-2\lambda_1} - 1)\} \sinh(\sqrt{-2\lambda_1}x), & 0 < x \le a \\ C_4 e^{-\sqrt{-2\lambda_1}x}\{e^{2\sqrt{-2\lambda_1}x} + e^{2\sqrt{-2\lambda_1}b}(\sqrt{-2\lambda_1} - 1)\} \sinh(\sqrt{-2\lambda_1}a), & a < x \le b \\ C_4\sqrt{-2\lambda_1} e^{\sqrt{-2\lambda_1}(2b - x)} \sinh(\sqrt{-2\lambda_1}a), & b < x, \end{cases}$$

where $C_4 = C_4(a, b, \lambda_1)$ is the positive normalizing constant. If we assume that $a = 1/4$, then $-2 < \lambda_1 < 0$ for $b > 1/4$.

6.2.2 In case of $0 < \alpha \le 2$

In this subsection, we assume that $0 < \alpha \le 2$.

Example 6.10. Suppose that $d = 1$ and $1 < \alpha \le 2$. Let $D = \mathbb{R}$ and $\mu = \sum_{i=1}^n \alpha_i \delta_{a_i}$, where $\alpha_i > 0$ and $-\infty < a_1 < a_2 < \cdots < a_n < \infty$. Denote by h the ground state corresponding to $\lambda_1(\alpha) := \lambda_1(\mu; \mathbb{R})$ with normalization $\int_{-\infty}^\infty h^2\, dx = 1$. Let $G_\beta(x, y), \beta > 0$, be the β-resolvent of $\mathbf{M}^\alpha$,

$$G_\beta(x, y) = \begin{cases} \dfrac{2^{1/\alpha}}{\pi} \displaystyle\int_0^\infty \frac{\cos\{2^{1/\alpha}(x - y)z\}}{\beta + z^\alpha}\, dz, & 1 < \alpha < 2 \\ \dfrac{1}{\sqrt{2\beta}} e^{-\sqrt{2\beta}|x - y|}, & \alpha = 2. \end{cases}$$

We then see in a similar way to Example 6.8 that

$$h(x) = \sum_{i=1}^n \alpha_i G_{-\lambda_1(\alpha)}(x, a_i) h(a_i)$$

and

$$\lambda_1(\alpha) = \min\{\kappa : |G_{-\kappa} - I| = 0\},$$

where G_β is the $n \times n$-matrix defined by $(\alpha_j G_\beta(a_i, a_j))_{1 \le i,j \le n}$. We now assume that $n = 1$, $a_1 = 0$ and $\alpha_1 = Q - 1 > 0$. For $1 < \alpha < 2$, since $(Q - 1)G_{-\lambda_1(\alpha)}(0, 0) = 1$ and

$$\begin{aligned} G_{-\lambda_1(a)}(0, 0) &= \frac{2^{1/\alpha}}{\pi(-\lambda_1(\alpha))^{(\alpha - 1)/\alpha}} \int_0^\infty \frac{1}{1 + z^\alpha}\, dz \\ &= \frac{2^{1/\alpha}}{\alpha \sin\left(\dfrac{\pi}{\alpha}\right)(-\lambda_1(\alpha))^{(\alpha - 1)/\alpha}}, \end{aligned}$$

it follows that

$$\lambda_1(\alpha) = -\left\{\frac{(Q - 1)2^{1/\alpha}}{\alpha \sin\left(\dfrac{\pi}{\alpha}\right)}\right\}^{\alpha/(\alpha - 1)}.$$

This value is also true for $\alpha = 2$. It also holds that

$$h(x) = \begin{cases} C \displaystyle\int_0^\infty \frac{\cos(2^{1/\alpha} x z)}{\lambda_1(\alpha) + z^\alpha} \, dz, & 1 < \alpha < 2 \\ (Q-1)^{1/2} \, e^{-(Q-1)|x|}, & \alpha = 2. \end{cases}$$

where $C = C(\alpha, Q)$ is the positive normalizing constant. The following is the graph of $\lambda_1(\alpha)$ for $1.4 < \alpha \leq 2$. We note that $\lim_{\alpha \downarrow 1} \lambda_1(\alpha) = -\infty$.

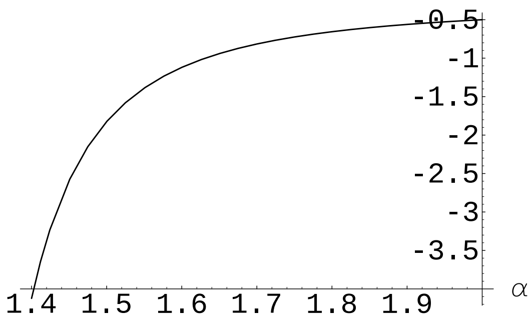

Figure 6.4: $\lambda_1(\alpha), 1.4 < \alpha \leq 2$

Appendix A

Positivity of the Green functions for symmetric α-stable processes

Let $\mathbf{M}^\alpha = (X_t, P_x)$ be the symmetric α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and $\mathbf{M}^D = (X_t^D, P_x^D)$ the absorbing α-stable process on an open set $D \subset \mathbb{R}^d$. Suppose that $\mathbf{M}^D$ is transient and denote its Green function by $G^D(x,y)$. In this appendix we prove

Theorem A.1. *For any open set $D \subset \mathbb{R}^d$, it holds that $G^D(x,y) > 0$ for any $x, y \in D$.*

We first show some lemmas needed for Theorem A.1. Denote by m the d-dimensional Lebesgue measure.

Lemma A.2. *For any closed set $F \subset D$ with $m(F) > 0$, it holds that $P_x^D(\sigma_F < \infty) > 0$ for any $x \in D$.*

Proof. Let $B(x,r) = \left\{y \in \mathbb{R}^d : |y - x| < r\right\}$ for $x \in D$ and $r > 0$. Set $\tilde{F} = F \cap B(x,r)^c$ and take $r > 0$ such that $m(\tilde{F}) > 0$. Then $\tilde{F} \cap B(x, r/2) = \emptyset$. By using the notion of the Lévy system (N^D, t) for $\mathbf{M}^D$ as defined in Chapter 1, it follows that for $x \in D$,

$$\begin{aligned} E_x^D\left[\sum_{s \le t} \mathbf{1}_{B(x,r/2)}(X_{s-})\mathbf{1}_{\tilde{F}}(X_s)\right] &= E_x^D\left[\int_0^t \int_{D_\Delta} \mathbf{1}_{B(x,r/2)}(X_s)\mathbf{1}_{\tilde{F}}(y) N^D(X_s, dy)\, ds\right] \\ &\ge E_x^D\left[\int_0^t \int_D \mathbf{1}_{B(x,r/2)}(X_s)\mathbf{1}_{\tilde{F}}(y) N^D(X_s, dy)\, ds\right] \\ &= \mathcal{A}(d,\alpha) E_x^D\left[\int_0^t \mathbf{1}_{B(x,r/2)}(X_s)\left(\int_D \frac{\mathbf{1}_{\tilde{F}}(y)}{|X_s - y|^{d+\alpha}}\, dy\right) ds\right]. \end{aligned} \tag{A.1}$$

Since $m(\tilde{F}) > 0$ implies that

$$\int_D \frac{\mathbf{1}_{\tilde{F}}(y)}{|x-y|^{d+\alpha}}\, dy > 0,\ x \in D,$$

it holds that

$$E_x^D\left[\sum_{s \le t} \mathbf{1}_{B(x,r/2)}(X_{s-})\mathbf{1}_{\tilde{F}}(X_s)\right] > 0.$$

Hence $P_x^D(\sigma_{\tilde{F}} \le t) > 0$, which implies that

$$\begin{aligned} P_x^D(\sigma_F < \infty) &\ge P_x^D(\sigma_{\tilde{F}} < \infty) \\ &\ge P_x^D(\sigma_{\tilde{F}} \le t) > 0, \quad x \in D. \end{aligned}$$

□

Let $G^D(x,K) = G^D\mathbf{1}_K(x) = \int_K G^D(x,y)\,dy$. We then have

Lemma A.3. *For any set $K \subset D$ with $m(K) > 0$, it holds that $G^D(x,K) > 0$ for any $x \in D$.*

Proof. It holds that

$$m\left(\{x \in D : G^D(x,K) > 0\}\right) > 0$$

for any set $K \subset D$ with $m(K) > 0$ because

$$\begin{aligned} \int_D G^D(x,K)\,dx &= \int_D G^D\mathbf{1}_K(x)\,dx \\ &= \int_D G^D\mathbf{1}(x)\,\mathbf{1}_K(x)\,dx > 0. \end{aligned}$$

Hence there exists a compact set $F \subset \{x \in D : G^D(x,K) > 0\}$ with $m(F) > 0$ such that $G^D(x,K) > 0$ for all $x \in F$. On the other hand, it follows that for $x \in D$,

$$\begin{aligned} G^D(x,K) &= E_x^D\left[\int_0^\infty \mathbf{1}_K(X_t)\,dt\right] \\ &\ge E_x^D\left[\int_{\sigma_F}^\infty \mathbf{1}_K(X_t)\,dt; \sigma_F < \infty\right]. \end{aligned}$$

Then the right hand side above is equal to

$$E_x^D\left[E_{X_{\sigma_F}}^D\left[\int_0^\infty \mathbf{1}_K(X_t)\,dt\right]; \sigma_F < \infty\right] = E_x^D\left[G^D(X_{\sigma_F},K); \sigma_F < \infty\right]$$

by the strong Markov property. Since $X_{\sigma_F} \in F$, we see from Lemma A.2 that

$$G^D(x,K) \ge E_x^D\left[G^D(X_{\sigma_F},K); \sigma_F < \infty\right] > 0, \quad x \in D.$$

□

Remark A.4. It follows that $G^D(x,y) = G^D(y,x) > 0$ for any $x \in D$ and m-a.e. $y \in D$ because the set K is arbitrary in Lemma A.3.

Proof of Theorem A.1. Denote by $p_t^D(x,y)$ the integral kernel of the Markovian transition semigroup of $\mathbf{M}^D$. Since

$$p_{t+s}^D(x,y) = \int_D p_t^D(x,z)\,p_s^D(z,y)\,dz,$$

it holds that

$$\begin{aligned} \int_t^\infty p_s^D(x,y)\,ds &= \int_D p_t^D(x,z)\left(\int_0^\infty p_s^D(z,y)\,ds\right)dz \\ &= \int_D p_t^D(x,z) G^D(z,y)\,dz. \end{aligned}$$

Because $\int_D p_t^D(x,y)\,dy > 0$, there exists a set $E \subset D$ such that $p_t^D(x,y) > 0$ for m-a.e. $y \in E$. Combining this with Remark A.4, we obtain

$$\begin{aligned} G^D(x,y) &\geq \int_t^\infty p_s^D(x,y)\,ds \\ &\geq \int_E p_t^D(x,z) G^D(z,y)\,dz > 0 \end{aligned}$$

for any $x, y \in D$. □

Remark A.5. Theorem 3.3 shows that the process $\mathbf{M}^D$ is irreducible for any open set $D \subset \mathbb{R}^d$ even if D is disconnected.

Bibliography

[1] S. Albeverio, P. Blanchard and Z. M. Ma, *Feynman-Kac semigroups in terms of signed smooth measures*, in "Random Partial Differential Equations" (U. Hornung et al. Eds.), Birkhäuser, Basel, 1991, pp. 1–31.

[2] S. Albeverio and Z. M. Ma, *Additive functionals, nowhere Radon and Kato class smooth measures associated with Dirichlet forms*, Osaka J. Math. **29** (1992), 247–265.

[3] J. P. Antoine, F. Gesztesy and J. Shabani, *Exact solvable models of sphere interactions in quantum mechanics*, J. Phys. A **20** (1987), 3687–3712.

[4] S. Asmussen and H. Hering, *Strong limit theorems for general supercritical branching processes with applications to branching diffusions*, Z. Wahrsch. Verw. Gebiete **36** (1976), 195–212.

[5] R. Bañuelos and T. Kulczycki, *The Cauchy process and Steklov problem*, J. Funct. Anal. **211** (2004), 355–423.

[6] R. Bañuelos, R. Latala and P. J. Méndez-Hernández, *A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes*, Proc. Amer. Math. Soc. **129** (2001), 2997–3008.

[7] A. Ben Amor, *Trace inequalities for operators associated to regular Dirichlet forms*, Forum Math. **16** (2004), 417–429.

[8] A. Benveniste and J. Jacod, *Syetems de Lévy des processus de Markov*, Invent. Math. **21** (1973), 183–198.

[9] R. M. Blumenthal and R. K. Getoor, Markov Processes and Potential Theory, Academic Press, New York, 1968.

[10] R. M. Blumenthal, R. K. Getoor and D. B. Ray, *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. **99** (1961), 540–554.

[11] A. N. Borodin and P. Salminen, Handbook of Brownian Motion – Facts and Formulae, Probability and Its Applications, Birkhäuser, Basel, 1996.

[12] R. Carmona, W. C. Masters and B. Simon, *Relativistic Schrödinger operators: Asymptotic behavior of the eigenfunctions*, J. Funct. Anal. **91** (1990), 117–142.

[13] M.-F. Chen, *The principal eigenvalue for jump processes*, Acta Math. Sin. (Engl. Ser.) **16** (2000), 361–368.

[14] Z.-Q. Chen, *Gaugeability and conditional gaugeability*, Trans. Amer. Math. Soc. **354** (2002), 4639–4679.

[15] Z.-Q. Chen, *Analytic characterization of conditional gaugeability for non-local Feynman-Kac transforms*, J. Funct. Anal. **202** (2003), 226–246.

[16] Z.-Q. Chen, P. J. Fitzsimmons, M. Takeda, J. Ying and T.-S. Zhang, *Absolute continuity of symmetric Markov processes*, Ann. Probab. **32** (2004), 2067–2098.

[17] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Proc. Appl. **108** (2003), 27–62.

[18] Z.-Q. Chen and R. Song, *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. **312** (1998), 465–501.

[19] Z.-Q. Chen and R. Song, *General gauge and conditional gauge theorems*, Ann. Probab. **30** (2002), 1313-1339.

[20] Z.-Q. Chen and R. Song, *Drift transforms and Green function estimates for discontinuous processes*, J. Funct. Anal. **201** (2003), 262–281.

[21] K. L. Chung and Z. X. Zhao, From Brownian Motion to Schrödinger's Equation, Springer-Verlag, Berlin, 1995.

[22] E. B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge, 1989.

[23] M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Wiener integrals for large time*, in "Proceedings of International Conference on Function Space" (A. M. Arthure Eds.), Oxford, 1974, pp. 15–33.

[24] M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time,* I, Comm. Pure Appl. Math. **28** (1975), 1–47.

[25] R. Durrett, Probability: Theory and Examples, 3rd ed., Duxbury Press, Belmony, 2004.

[26] J. Engländer and A. E. Kyprianou, *Local extinction versus local exponential growth for spatial branching processes*, Ann. Probab. **32** (2004), 78–99.

[27] J. Engländer and R. G. Pinsky, *On the construction and support properties of measure-valued diffusions on $D \subseteq \mathbb{R}^d$ with spatially dependent branching*, Ann. Probab. **27** (1999), 684–730.

[28] P. J. Fitzsimmons, *Hardy's inequality for Dirichlet forms*, J. Math. Anal. Appl. **250** (2000), 548–560.

[29] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin, 1994.

[30] R. K. Getoor, *First passage times for symmetric stable processes in space*, Trans. Amer. Math. Soc. **101** (1961), 75–90.

[31] R. K. Getoor, *Continuous additive functionals of a Markov process with applications to processes with independent increments*, J. Math. Anal. Appl. **13** (1966), 132–153.

[32] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[33] S. W. He, J. G. Wang and J. A. Yan, Semimartingale Theory and Stochastic Calculus, Science Press, Beijing, 1992.

[34] N. Ikeda, M. Nagasawa and S. Watanabe, *Branching Markov processes* I, J. Math. Kyoto Univ. **8** (1968), 233–278 .

[35] N. Ikeda, M. Nagasawa and S. Watanabe, *Branching Markov processes* II, J. Math. Kyoto Univ. **8** (1968), 365–410.

[36] F. I. Karpelevich, E. A. Pechersky and Y. M. Suhov, *A phase transition for hyperbolic branching processes*, Comm. Math. Phys. **195** (1998), 627–642.

[37] M. Kac, *Some connections between probability theory and differential and integral equations*, in "Proceedings of the second Berkeley Symposium on Mathematical Statistics and Probability, 1950", Univ. California Press, Berkeley and Los Angels, 1951, pp.189–215.

[38] R. Z. Khas'minskii, *On positive solutions of the equation* $\mathcal{U}u + Vu = 0$, Theory Probab. Appl. **4** (1959), 309–318.

[39] S. P. Lalley and T. Sellke, *Hyperbolic branching Brownian motion*, Probab. Theory Related Fields **108** (1997), 171–192.

[40] V. G. Maz'ja, Sobolev Spaces, Springer, Berlin, 1985.

[41] Y. Ogura, *A limit theorem for particle numbers in bounded domains of a branching diffusion process*, Stoch. Proc. Appl. **14** (1983), 19–40.

[42] R. G. Pinsky, Positive Harmonic Functions and Diffusion, Cambridge Univ. Press, Cambridge, 1995.

[43] R. G. Pinsky, *Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions*, Ann. Probab. **24** (1996), 237–267.

[44] S. C. Port, *Hitting times and potentials for recurrent stable processes*, J. Anal. Math. **20** (1967), 371–395.

[45] D. Ray, *Stable processes with an absorbing barrier*, Trans. Amer. Math. Soc. **87** (1958), 187–197.

[46] D. Revuz and M. Yor, Continuous Martingales and Brownian Motion, 3rd ed., Springer-Verlag, Berlin, 1999.

[47] L. C. G. Rogers and D. Williams, Diffusion Processes, Markov Processes, and Martingales, Vol. 1: Foundations, 2nd edn, Wiley, Chichester, 1994.

[48] S. Sato, *An inequality for the spectral radius of Markov processes*, Kodai Math. J. **8** (1985), 5–13.

[49] B. A. Sevast'yanov, *Branching stochastic processes for particles diffusing in a bounded domain with absorbing boundaries*, Theory Probab. Appl. **3** (1958), 111–126.

[50] P. Stollmann and J. Voigt, *Perturbation of Dirichlet forms by measures*, Potential Anal. **5** (1996), 109–138.

[51] M. Takeda, *On a large deviation for symmetric Markov processes with finite life time*, Stoch. Stoch. Rep. **59** (1996), 143–167.

[52] M. Takeda, *Conditional gaugeability and subcriticality of generalized Schrödinger operators*, J. Funct. Anal. **191** (2002), 343–376.

[53] M. Takeda, *Large deviation principle for additive functionals of Brownian motion corresponding to Kato class measures*, Potential Anal. **19** (2003), 51–67.

[54] M. Takeda, *Gaugeability for Feynman-Kac functionals with applications to symmetric α-stable processes*, Proc. Amer. Math. Soc. **134** (2006), 2729–2738.

[55] M. Takeda, *Large deviations for additive functionals of symmetric stable processes*, preprint.

[56] M. Takeda, *Branching Brownian motion on Riemannian manifolds: Expectation of the number of branches hitting closed sets*, to appear in Potential Anal.

[57] M. Takeda and K. Tsuchida, *Differentiability of spectral functions for symmetric α-stable processes*, to appear in Trans. Amer. Math. Soc.

[58] M. Takeda and T. Uemura, *Subcriticality and gaugeability for symmetric α-stable processes*, Forum Math. **16** (2004), 505–517.

[59] Z. Vondraček, *An estimate for the L^2-norm of a quasi continuous function with respect to a smooth measure*, Arch. Math. **67** (1996), 408–414.

[60] S. Watanabe, *On stable processes with boundary conditions*, J. Math. Soc. Japan. **14** (1962), 170–198.

[61] S. Watanabe, *On the branching process for Brownian particles with an absorbing boundary*, J. Math. Kyoto Univ. **4** (1965), 385–398.

[62] S. Watanabe, *Limit theorems for a class of branching processes*, in " Markov Processes and Potential Theory" (J. Chover Eds.), Wiley, New York, 1967, pp. 205–232.

[63] S. Watanabe, Stochastic Differential Equations, Sangyôtosho, 1975 (in Japanese).