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Berry–Esseen bound for the Brownian motions on hyperbolic spaces

by

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Abstract. We obtain the uniform convergence rate for the Gaussian fluctuation of the radial part of the Brownian motion on a hyperbolic space. We also show that this result is sharp if the dimension of the hyperbolic space is 2 or general odd. Our approach is based on the repetitive use of the Millson formula and the integration by parts formula.

1. Introduction. We are concerned with the Gaussian fluctuation of the radial part of the Brownian motion on a d-dimensional hyperbolic space. In this note, we obtain the uniform convergence rate for the fluctuation in distribution, the so-called Berry–Esseen bound, together with the sharpness for $d = 2$ and general odd $d \geq 3$.

For $d \geq 2$, let $\mathbb{H}^d$ be the d-dimensional hyperbolic space with a pole o, and let $d = d_{\mathbb{H}^d}$ be the associated distance function. Let $\Delta = \Delta_{\mathbb{H}^d}$ be the Laplace–Beltrami operator, and $X = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{H}^d})$ the Brownian motion on $\mathbb{H}^d$ generated by $\Delta/2$. Let $R_t^{(d)} = d(o, X_t)$ $(t \geq 0)$ be the radial process and $P = P_o$. Then by the Itô formula applied to $R_t^{(d)}$, we have

$$R_t^{(d)} = B_t + \frac{d-1}{2}\int_0^t \coth R_s^{(d)}\,\mathrm{d}s, \tag{1.1}$$

where B_t is the Brownian motion on $\mathbb{R}$ (see, e.g., [11, Example 3.3.3]). Since $\lim_{t\to\infty} R_t^{(d)} = \infty$, we obtain the law of large numbers:

$$\lim_{t\to\infty} \frac{R_t^{(d)}}{t} = \frac{d-1}{2}, \quad P\text{-a.s.} \tag{1.2}$$

(see also [7, Section 4.1] and [14] for the escape rate). Moreover, we realize

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the limiting behavior of the fluctuation in (1.2) as the central limit theorem:

$$\lim_{t\to\infty} P\left(\frac{R_t^{(d)}-(d-1)t/2}{\sqrt{t}}\geq x\right)=\Phi(x),\qquad x\in\mathbb{R} \tag{1.3}$$

(see, e.g., [2, Corollary 3.1] and [13, Theorem 2.1]), where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-u^2/2}\,\mathrm{d}u,\qquad x\in\mathbb{R}.$$

In connection with the heat equation in $\mathbb{H}^d$, Vázquez [15, (5.5)] noted that the limit in (1.3) is uniform in $x\in\mathbb{R}$. Our purpose in this note is to establish the uniform convergence rate in (1.3):

Theorem 1.1. *For any $d\geq 2$, there exists a constant $c_1>0$ such that*

$$\sup_{x\in\mathbb{R}}\left|P\left(\frac{R_t^{(d)}-(d-1)t/2}{\sqrt{t}}\geq x\right)-\Phi(x)\right|\leq\frac{c_1}{\sqrt{t}},\qquad t\geq 1. \tag{1.4}$$

Moreover, if $d=2$ or if $d\geq 3$ is odd, then there exists a constant $c_2>0$ such that

$$P\left(\frac{R_t^{(d)}-(d-1)t/2}{\sqrt{t}}\geq 0\right)-\Phi(0)\geq\frac{c_2}{\sqrt{t}},\qquad t\geq 1. \tag{1.5}$$

That is, in these cases, the convergence rate of (1.4) *is sharp.*

Theorem 1.1 provides the convergence rate $t^{-1/2}$, which is consistent with the standard Berry–Esseen theorem for i.i.d. random variables (see, e.g., [6, Theorem 3.4.17]). Note that for $d=1$, X is the Brownian motion on $\mathbb{R}$ started at the origin. Since the radial process $R_t^{(1)}$ is the reflecting Brownian motion, for any $t>0$ we have

$$P\left(\frac{R_t^{(1)}}{\sqrt{t}}\geq x\right)=\begin{cases}1 & (x\leq 0),\\ 2\Phi(x) & (x>0).\end{cases}$$

In other words, it is unnecessary to study the limiting behavior of the fluctuation as in (1.3).

To prove Theorem 1.1, we make repetitive use of the Millson formula (see (2.4) below) together with the integration by parts formula. By the Millson formula, we reduce the calculation of the distribution under consideration to $d=2$ or $d=3$. At present, the validity of (1.5) is unavailable for even $d\geq 4$ because of implicit constants in the calculation (see Remark 4.3 below for details).

Even though we know the matching bound on the transition density function of X (see (2.5) below), this bound is insufficient for the estimate of the distribution which would be necessary in the proof of (1.4). According to (1.1) and (1.2), we can also regard $R_t^{(d)}$ as the Brownian motion with linear drift asymptotically. Hence we might guess that the Girsanov theorem (see, e.g., [12, Theorem 5.1 (p. 191) and Corollary 5.13 (p. 199)]) is effective for (1.1); however, as the so-called Novikov condition fails (see (2.6) below), it is unclear whether the Girsanov theorem is applicable to (1.1). Since no information is available about the convergence rate in distribution of (1.2), we do not know whether the expression (1.1) is applicable for the proof of (1.4).

Concerning the law of large numbers (1.2), Cammarota–De Gregorio–Macci [3, Proposition 3.2] and Hirao [10, Theorem 1.1] proved the large deviation principle for the radial part by using (2.5). Moreover, Cammarota–De Gregorio–Macci [3, Section 3, pp. 1560–1563] proved the moderate deviation principle, together with the exponential decay order of the distribution related to asymptotic normality. Theorem 1.1 establishes the asymptotic normality in the suitable scaling factor.

Our motivation lies in clarifying how the structures of the volume and spectrum determine the long time behavior of a symmetric Markov process. Anker–Setti [1, Theorem 2] revealed the concentration behavior in distribution of the Brownian motion on a complete and non-compact Riemannian manifold for which the volume growth rate is exponential and the bottom of the L^2-spectrum of the Laplacian is strictly positive. In particular, this result characterizes the linear growth rate of the radial part of the Brownian motion in terms of the exponential volume growth rate and the bottom of the L^2-spectrum. For a Riemannian manifold with a pole, Grigor'yan–Hsu [7, Theorem 4.1] determined the linear growth rate exactly. Theorem 1.1 is an attempt to provide the second order asymptotic behavior of the linear growth phenomena for a symmetric Markov process in a quantitative way.

We close this Introduction with some words on the contents and notation. In Section 2, we collect basic facts about the Brownian motions on hyperbolic spaces. In Sections 3 and 4, we prove Theorem 1.1 for odd dimensions and even dimensions, respectively. Two lemmas in elementary calculus are postponed to the Appendix.

For a fixed constant $T > 0$, let $f_1(t)$ and $f_2(t)$ be positive functions defined on $[T, \infty)$. We then write $f_1(t) \sim f_2(t)$ if $f_1(t)/f_2(t) \to 1$ as $t \to \infty$. Let S be a set, and let $g_1(s)$ and $g_2(s)$ be positive functions defined on S. We then write $g_1(s) \lesssim g_2(s)$ if there exists a constant $c > 0$ such that $g_1(s) \le c g_2(s)$ for all $s \in S$. We further write $g_1(s) \asymp g_2(s)$ if $g_1(s) \lesssim g_2(s)$ and $g_2(s) \lesssim g_1(s)$. We also make the conventions $\sum_{n=1}^{0} = 0$ and $\prod_{n=1}^{0} = 1$.

2. Preliminaries. In this section, we recall the Millson formula and the estimates of the transition density function of the Brownian motion on $\mathbb{H}^d$.

Let $d \geq 2$. The d-dimensional hyperbolic space $\mathbb{H}^d$ is a spherically symmetric Riemannian manifold with the Riemannian distance given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + (\sinh r)^2\,\mathrm{d}\theta^2.$$

Here $\mathrm{d}\theta^2$ is the distance on the $(d-1)$-dimensional surface $S^{d-1} = \{\theta \in \mathbb{R}^d \mid |\theta| = 1\}$. We write $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ for the surface area of S^{d-1}. Let d and $\mathrm{d}v$ denote the associated distance function and volume measure, respectively.

Let $X = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{H}^d})$ be the Brownian motion on $\mathbb{H}^d$ generated by $\Delta/2$, half of the Laplace–Beltrami operator on $\mathbb{H}^d$. Then there exists a Borel measurable function $p_d(t,x,y) : (0,\infty)\times\mathbb{H}^d\times\mathbb{H}^d \to (0,\infty)$ such that

$$P_x(X_t \in A) = \int_A p_d(t,x,y)\,v(\mathrm{d}y), \qquad x\in\mathbb{H}^d,\ t>0,\ A\in\mathcal{B}(\mathbb{H}^d).$$

Namely, $p_d(t,x,y)$ is the transition density function of X. Moreover, there exists a Borel measurable function $q_d(t,r) : (0,\infty)\times(0,\infty)\to(0,\infty)$ such that $p_d(t,x,y) = q_d(t,d(x,y))$ for any $x,y\in\mathbb{H}^d$ and $t>0$, and

$$P_x(d(x,X_t)\in B) = \omega_d\int_B q_d(t,r)\sinh^{d-1} r\,\mathrm{d}r, \qquad t>0,\ B\in\mathcal{B}([0,\infty)) \tag{2.1}$$

(see, e.g., [3, §2.2] for details).

It is known that

$$q_2(t,r) = \frac{2^{1/2}e^{-t/8}}{(2\pi t)^{3/2}}\int_r^\infty \frac{se^{-s^2/(2t)}}{(\cosh s-\cosh r)^{1/2}}\,\mathrm{d}s, \tag{2.2}$$

$$q_3(t,r) = \frac{e^{-t/2}}{(2\pi t)^{3/2}}\,\frac{r}{\sinh r}e^{-r^2/(2t)} \tag{2.3}$$

(see, e.g., [4, Section 5.7] or [9, Section 2] and references therein). The *Millson formula* is a recursive relation between the transition density functions:

$$q_d(t,r) = -\frac{e^{-(d-2)t/2}}{2\pi\sinh r}\,\frac{\partial q_{d-2}}{\partial r}(t,r), \qquad t>0,\ r>0 \tag{2.4}$$

(see, e.g., [5, 8], [4, Section 5.7] and references therein). We also see by [4, Theorem 5.7.2] that

$$q_d(t,r) \asymp \frac{1}{t^{d/2}}\exp\left(-\frac{(d-1)^2}{8}t - \frac{d-1}{2}r - \frac{r^2}{2t}\right) \times (1+r+t)^{(d-3)/2}(1+r), \qquad t>0,\ r>0. \tag{2.5}$$

Remark 2.1. As mentioned in the Introduction, the Novikov condition fails for (1.1), that is,

$$E\left[\exp\left(\frac{1}{2}\int_0^t \coth^2 R_s^{(d)}\,\mathrm{d}s\right)\right] = \infty, \qquad t>0. \tag{2.6}$$

To verify this equality, it is enough to show that

$$E\Big[\int_0^t \coth^2 R_s^{(d)}\,\mathrm{d}s\Big] = \infty, \qquad t>0. \tag{2.7}$$

We show it by using (2.5). Fix $t>0$. Then

$$\begin{aligned} E\Big[\int_0^t \coth^2 R_s^{(d)}\,\mathrm{d}s\Big] &= \int_0^t E[\coth^2 R_s^{(d)}]\,\mathrm{d}s \\ &= \omega_d \int_0^t \Big(\int_0^\infty (\coth^2 r) q_d(s,r) \sinh^{d-1} r\,\mathrm{d}r\Big)\,\mathrm{d}s = \omega_d I(t). \end{aligned}$$

By (2.5), there exists a constant $c_1 = c_1(t) > 0$ such that for any $s\in[0,t]$ and $r\in[0,\sqrt{s}]$,

$$q_d(s,r) \ge \frac{c_1 e^{-r^2/(2s)}}{s^{d/2}}.$$

By combining this with

$$\sinh r \sim r, \quad \coth r \sim \frac{1}{r} \quad (r\to 0),$$

there exists a constant $c_2 = c_2(t) > 0$ such that for any $s\in[0,t]$ and $r\in[0,\sqrt{s}]$,

$$(\coth^2 r) q_d(s,r)\sinh^{d-1} r \ge \frac{c_2}{r^2}\,\frac{e^{-r^2/(2s)}}{s^{d/2}} r^{d-1} = \frac{c_2 e^{-r^2/(2s)}}{s^{d/2}} r^{d-3}.$$

This implies that

$$\begin{aligned} I(t) &\ge \int_0^t \Big(\int_0^{\sqrt{s}} (\coth^2 r) q_d(s,r)\sinh^{d-1} r\,\mathrm{d}r\Big)\,\mathrm{d}s \\ &\ge c_2 \int_0^t \Big(\int_0^{\sqrt{s}} e^{-r^2/(2s)} r^{d-3}\,\mathrm{d}r\Big)\frac{1}{s^{d/2}}\,\mathrm{d}s \\ &= c_2 \int_0^t \frac{1}{s}\,\mathrm{d}s \Big(\int_0^1 e^{-u^2/2} u^{d-3}\,\mathrm{d}u\Big) = \infty. \end{aligned}$$

At the first equality above, we used the change of variables formula with $r=\sqrt{s}\,u$. We thus get (2.7).

Before the proof of Theorem 1.1 in the subsequent sections, we make a comment on the range of $x\in\mathbb{R}$ in (1.4). Since $R_t^{(d)} \ge 0$ for any $t\ge 0$, we have, for any $x \le -(d-1)\sqrt{t}/2$,

$$P\Big(\frac{R_t^{(d)} - (d-1)t/2}{\sqrt{t}} \ge x\Big) = 1,$$

which implies that

$$\left|P\left(\frac{R_t^{(d)}-(d-1)t/2}{\sqrt{t}}\geq x\right)-\Phi(x)\right| = 1-\Phi(x)$$
$$\leq 1-\Phi\left(-\frac{d-1}{2}\sqrt{t}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-(d-1)\sqrt{t}/2} e^{-y^2/2}\,\mathrm{d}y.$$

We also note that for $u>0$,

$$\int_{-\infty}^{-u} e^{-y^2/2}\,\mathrm{d}y = \int_u^{\infty} e^{-y^2/2}\,\mathrm{d}y = \int_u^{\infty}\frac{1}{y}ye^{-y^2/2}\,\mathrm{d}y \leq \frac{1}{u}\int_u^{\infty} ye^{-y^2/2}\,\mathrm{d}y = \frac{e^{-u^2/2}}{u}, \tag{2.8}$$

which yields

$$\int_{-\infty}^{-(d-1)\sqrt{t}/2} e^{-y^2/2}\,\mathrm{d}y \leq \frac{2e^{-(d-1)^2t/8}}{(d-1)\sqrt{t}}.$$

Then there exists a constant $c_1>0$ such that for any $t\geq 1$,

$$\sup_{x\leq-(d-1)\sqrt{t}/2}\left|P\left(\frac{R_t^{(d)}-(d-1)t/2}{\sqrt{t}}\geq x\right)-\Phi(x)\right| \leq \frac{c_1e^{-(d-1)^2t/8}}{\sqrt{t}} \leq \frac{c_1}{\sqrt{t}}.$$

Hence for the proof of (1.4) it is enough to show that, for some $c_2>0$,

$$\sup_{x\geq-(d-1)\sqrt{t}/2}\left|P\left(\frac{R_t^{(d)}-(d-1)t/2}{\sqrt{t}}\geq x\right)-\Phi(x)\right| \leq \frac{c_2}{\sqrt{t}}, \qquad t\geq 1. \tag{2.9}$$

3. Proof of Theorem 1.1 for odd dimensions. In this section, we first prove Theorem 1.1 for $d=3$. Using this assertion, we next prove the theorem for general odd dimensions.

3.1. Proof of Theorem 1.1 for $d=3$. We first prove (2.9) for $d=3$. We write $R_t=R_t^{(3)}$ for simplicity. Since $R_t\geq 0$ for any $t\geq 0$ and $\omega_3=4\pi$, it follows by (2.3) that for any $x\geq-\sqrt{t}$,

$$P\left(\frac{R_t-t}{\sqrt{t}}\geq x\right) = P(R_t\geq t+x\sqrt{t})$$
$$= \frac{\omega_3}{(2\pi t)^{3/2}}\int_{t+x\sqrt{t}}^{\infty} e^{-t/2}e^{-u^2/(2t)}u\sinh u\,\mathrm{d}u$$
$$= \frac{1}{\sqrt{2\pi}}\frac{2e^{-t/2}}{t\sqrt{t}}\int_{t+x\sqrt{t}}^{\infty} e^{-u^2/(2t)}u\sinh u\,\mathrm{d}u.$$

Then by the change of variables $u = t + v\sqrt{t}$ we obtain

$$\begin{aligned}\frac{2e^{-t/2}}{t\sqrt{t}} \int_{t+x\sqrt{t}}^{\infty} & e^{-u^2/(2t)} u \sinh u \,\mathrm{d}u \\ &= 2e^{-t} \int_x^{\infty} e^{-v^2/2} e^{-v\sqrt{t}} \sinh(t + v\sqrt{t}) \,\mathrm{d}v \\ &\quad + \frac{2e^{-t}}{\sqrt{t}} \int_x^{\infty} e^{-v^2/2} e^{-v\sqrt{t}} v \sinh(t + v\sqrt{t}) \,\mathrm{d}v \\ &= I_1(t,x) + I_2(t,x).\end{aligned}$$

Therefore,

$$P\left(\frac{R_t - t}{\sqrt{t}} \geq x\right) - \Phi(x) = \frac{1}{\sqrt{2\pi}} \left\{ \left(I_1(t,x) - \int_x^{\infty} e^{-v^2/2} \,\mathrm{d}v \right) + I_2(t,x) \right\}. \tag{3.1}$$

We have

$$\begin{aligned} I_1(t,x) - \int_x^{\infty} e^{-v^2/2} \,\mathrm{d}v &= \int_x^{\infty} e^{-v^2/2} (1 - e^{-2(t+v\sqrt{t})}) \,\mathrm{d}v - \int_x^{\infty} e^{-v^2/2} \,\mathrm{d}v \\ &= -\int_x^{\infty} e^{-v^2/2} e^{-2(t+v\sqrt{t})} \,\mathrm{d}v. \end{aligned} \tag{3.2}$$

Then by the change of variables $u = v + 2\sqrt{t}$ we get, for any $t \geq 1$ and $x \geq -\sqrt{t}$,

$$\int_x^{\infty} e^{-v^2/2} e^{-2(t+v\sqrt{t})} \,\mathrm{d}v = \int_{x+2\sqrt{t}}^{\infty} e^{-u^2/2} \,\mathrm{d}u \leq \int_{\sqrt{t}}^{\infty} e^{-u^2/2} \,\mathrm{d}u \leq \frac{e^{-t/2}}{\sqrt{t}}. \tag{3.3}$$

In the last inequality above, we have used (2.8). Hence for any $t \geq 1$ and $x \geq -\sqrt{t}$,

$$0 \geq I_1(t,x) - \int_x^{\infty} e^{-u^2/2} \,\mathrm{d}u \geq -\frac{e^{-t/2}}{\sqrt{t}}. \tag{3.4}$$

Since $(e^{-v^2/2})' = -ve^{-v^2/2}$, by integration by parts we have

$$\begin{aligned} I_2(t,x) &= \frac{1}{\sqrt{t}} \int_x^{\infty} v e^{-v^2/2} (1 - e^{-2(t+v\sqrt{t})}) \,\mathrm{d}v \\ &= \frac{e^{-x^2/2}}{\sqrt{t}} (1 - e^{-2(t+x\sqrt{t})}) + 2 \int_x^{\infty} e^{-v^2/2} e^{-2(t+v\sqrt{t})} \,\mathrm{d}v \geq 0. \end{aligned} \tag{3.5}$$

In particular, we see by (3.3) that for any $t \geq 1$ and $x \geq -\sqrt{t}$,

$$0 \leq I_2(t,x) \leq \frac{e^{-x^2/2}}{\sqrt{t}} + \frac{2e^{-t/2}}{\sqrt{t}} \leq \frac{3}{\sqrt{t}}.$$

Combining this with (3.1) and (3.4), we get (2.9) for $d = 3$.

We next prove (1.5) for $d = 3$. If we take $x = 0$ in (3.1), then by a calculation similar to (3.2) and (3.3), it follows that

$$I_1(t,0) - \int_0^\infty e^{-u^2/2}\,\mathrm{d}u = -\int_{2\sqrt{t}}^\infty e^{-u^2/2}\,\mathrm{d}u \geq -\frac{e^{-2t}}{2\sqrt{t}} \geq -\frac{e^{-2}}{2\sqrt{t}}, \qquad t \geq 1.$$

We also have, for any $t \geq 1$,

$$\begin{aligned} I_2(t,0) &= \frac{1}{\sqrt{t}} \int_0^\infty v e^{-v^2/2}(1 - e^{-2(t+v\sqrt{t})})\,\mathrm{d}v \\ &\geq \frac{1-e^{-2}}{\sqrt{t}} \int_0^\infty v e^{-v^2/2}\,\mathrm{d}v = \frac{1-e^{-2}}{\sqrt{t}}. \end{aligned}$$

Hence by (3.1) with $x = 0$, for any $t \geq 1$ we get

$$\begin{aligned} P\Big(\frac{R_t - t}{\sqrt{t}} \geq 0\Big) - \Phi(0) &= \frac{1}{\sqrt{2\pi}}\Big\{\Big(I_1(t,0) - \int_0^\infty e^{-u^2/2}\,\mathrm{d}u\Big) + I_2(t,0)\Big\} \\ &\geq \frac{1}{\sqrt{2\pi}}\Big(-\frac{e^{-2}}{2\sqrt{t}} + \frac{1-e^{-2}}{\sqrt{t}}\Big) = \frac{2-3e^{-2}}{2\sqrt{2\pi}}\,\frac{1}{\sqrt{t}}, \end{aligned}$$

which implies (1.5) for $d = 3$.

3.2. Proof of (1.4) for odd dimensions. Let $d \geq 2$. For $t \geq 1$ and $x \geq -(d-1)\sqrt{t}/2$, let

$$T = T(t,x) = x\sqrt{t} + \frac{d-1}{2}t \quad (\geq 0).$$

Since $R_t^{(d)} \geq 0$, by (2.1) and (2.4) we have

$$\begin{aligned} P\Big(\frac{R_t^{(d)} - (d-1)t/2}{\sqrt{t}} \geq x\Big) &= \omega_d \int_T^\infty q_d(t,r) \sinh^{d-1} r\,\mathrm{d}r \\ &= -\frac{\omega_d e^{-(d-2)t/2}}{2\pi} \int_T^\infty \frac{\partial q_{d-2}}{\partial r}(t,r) \sinh^{d-2} r\,\mathrm{d}r. \end{aligned} \tag{3.6}$$

Then by integration by parts and (2.5),

$$
\begin{aligned}
&-\int_T^\infty \frac{\partial q_{d-2}}{\partial r}(t,r)\sinh^{d-2} r\,\mathrm{d}r \\
&\qquad = [-q_{d-2}(t,r)\sinh^{d-2} r]_{r=T}^{r=\infty} + \int_T^\infty q_{d-2}(t,r)\Big(\frac{\partial}{\partial r}\sinh^{d-2} r\Big)\,\mathrm{d}r \\
&\qquad = q_{d-2}(t,T)\sinh^{d-2} T + \int_T^\infty q_{d-2}(t,r)\Big(\frac{\partial}{\partial r}\sinh^{d-2} r\Big)\,\mathrm{d}r.
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
&\int_T^\infty q_{d-2}(t,r)\Big(\frac{\partial}{\partial r}\sinh^{d-2} r\Big)\,\mathrm{d}r \\
&\qquad = -\frac{e^{-(d-4)t/2}}{2\pi}\int_T^\infty \frac{1}{\sinh r}\,\frac{\partial q_{d-4}}{\partial r}(t,r)\Big(\frac{\partial}{\partial r}\sinh^{d-2} r\Big)\,\mathrm{d}r \\
&\qquad = \frac{e^{-(d-4)t/2}}{2\pi} q_{d-4}(t,T)\Big(\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)\sinh^{d-2} r\Big|_{r=T} \\
&\qquad\quad + \frac{e^{-(d-4)t/2}}{2\pi}\int_T^\infty q_{d-4}(t,r)\frac{\partial}{\partial r}\Big(\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)\sinh^{d-2} r\,\mathrm{d}r.
\end{aligned}
$$

Repeating this procedure, for any $n \geq 2$ with $d \geq 2n$ we see from (3.6) that

(3.7)

$$
\begin{aligned}
&P\Big(\frac{R_t^{(d)} - (d-1)t/2}{\sqrt{t}} \geq x\Big) \\
&= \omega_d \sum_{m=1}^{n-1}\prod_{j=1}^{m}\frac{e^{-(d-2j)t/2}}{2\pi} q_{d-2m}(t,T)\Big(\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)^{m-1}\sinh^{d-2} r\Big|_{r=T} \\
&\quad + \omega_d \prod_{j=1}^{n-1}\frac{e^{-(d-2j)t/2}}{2\pi}\int_T^\infty\Big(\Big(\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)^{n-1}\sinh^{d-2} r\Big) q_{d-2n+2}(t,r)\sinh r\,\mathrm{d}r \\
&= \omega_d \sum_{m=1}^{n-1}\frac{e^{-(d-(m+1))mt/2}}{(2\pi)^m} q_{d-2m}(t,T)\Big(\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)^{m-1}\sinh^{d-2} r\Big|_{r=T} \\
&\quad + \omega_d \frac{e^{-(d-n)(n-1)t/2}}{(2\pi)^{n-1}}\int_T^\infty\Big(\Big(\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)^{n-1}\sinh^{d-2} r\Big) q_{d-2n+2}(t,r)\sinh r\,\mathrm{d}r.
\end{aligned}
$$

We now define $J_1^{(2n+1)}(t,x)$ and $J_2^{(2n+1)}(t,x)$ by $(3.7)_2$ and $(3.7)_4$, respectively, with $d=2n+1$:

$$J_1^{(2n+1)}(t,x) = \omega_{2n+1}\sum_{m=1}^{n-1}\frac{e^{-(2n-m)mt/2}}{(2\pi)^m}q_{2n+1-2m}(t,T)\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m-1}\sinh^{2n-1}r\bigg|_{r=T}, \tag{3.8}$$

$$J_2^{(2n+1)}(t,x) = \omega_{2n+1}\frac{e^{-(n^2-1)t/2}}{(2\pi)^{n-1}}\int_T^\infty\left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-1}\sinh^{2n-1}r\right)q_3(t,r)\sinh r\,\mathrm{d}r. \tag{3.9}$$

Then

$$P\left(\frac{R_t^{(2n+1)}-nt/2}{\sqrt{t}}\geq x\right)-\Phi(x)=J_1^{(2n+1)}(t,x)+(J_2^{(2n+1)}(t,x)-\Phi(x)). \tag{3.10}$$

Let us give an upper bound of $J_1^{(2n+1)}(t,x)$.

Lemma 3.1. *There exists a constant $c>0$ such that*

$$\sup_{x\geq -n\sqrt{t}} J_1^{(2n+1)}(t,x)\leq\frac{c}{\sqrt{t}},\qquad t\geq 1. \tag{3.11}$$

Proof. We give an upper bound of $e^{-(d-m-1)mt/2}q_{d-2m}(t,T)$; recall that $d=2n+1$. For $m=1,\dots,n-1$, we see by (2.5) that

$$\begin{aligned} e^{-(d-m-1)mt/2}&q_{d-2m}(t,T) \\ &\asymp \frac{1}{t^{(d-2m)/2}}e^{-(d-m-1)mt/2} \\ &\quad\times\exp\left(-\frac{(d-2m-1)^2}{8}t-\frac{(d-2m-1)T}{2}-\frac{T^2}{2t}\right) \\ &\quad\times(1+T+t)^{(d-2m-3)/2}(1+T). \end{aligned} \tag{3.12}$$

Then

$$e^{-(d-m-1)mt/2}\exp\left(-\frac{(d-2m-1)^2}{8}t-\frac{(d-2m-1)T}{2}-\frac{T^2}{2t}\right) = e^{-(d-m-1)T}e^{-x^2/2}. \tag{3.13}$$

We can also see that there exists a constant $c_1>0$ such that for any $t\geq 1$

and $x \geq -(d-1)\sqrt{t}/2$,

$$\begin{aligned}(1+T+t)^{(d-2m-3)/2}(1+T)& \\ &\leq (1+T+t)^{(d-2m-1)/2} = (1+x\sqrt{t}+(d-1)t/2+t)^{(d-2m-1)/2} \\ &\leq c_1(1+(|x|\sqrt{t})^{(d-2m-1)/2}+t^{(d-2m-1)/2}).\end{aligned}$$

Hence

$$\begin{aligned}&e^{-(d-m-1)mt/2}q_{d-2m}(t,T) \\ &\leq \frac{c_2}{t^{(d-2m)/2}}e^{-(d-m-1)T}e^{-x^2/2}\big(1+(|x|\sqrt{t})^{(d-2m-1)/2}+t^{(d-2m-1)/2}\big) \\ &\leq c_3 e^{-(d-m-1)T}\left(\frac{1}{t^{(d-2m)/2}}+\frac{e^{-x^2/2}|x|^{(d-2m-1)/2}}{t^{(d-2m)/4+1/4}}+\frac{1}{\sqrt{t}}\right) \leq \frac{c_4 e^{-(d-m-1)T}}{\sqrt{t}}.\end{aligned} \tag{3.14}$$

At the last inequality, we used the fact that $\sup_{x\in\mathbb{R}} e^{-x^2/2}|x|^{(d-2m-1)/2} < \infty$.

By (A.6), for any m $(1 \leq m \leq n-1)$, there exists a constant $c_5 > 0$ such that

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m-1}\sinh^{d-2} r\bigg|_{r=T} \leq c_5 e^{(d-m-1)T}.$$

By combining this with (3.14), there exists a constant $c_6 > 0$ such that for any $t \geq 1$ and $x \geq -(d-1)\sqrt{t}/2$,

$$\begin{aligned}&\frac{e^{-(d-m-1)mt/2}}{(2\pi)^m}q_{d-2m}(t,T)\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m-1}\sinh^{d-2} r\bigg|_{r=T} \\ &\qquad \leq \frac{c_6 e^{-(d-m-1)T}}{\sqrt{t}}\cdot e^{(d-m-1)T} = \frac{c_6}{\sqrt{t}}.\end{aligned} \tag{3.15}$$

Therefore, by the definition of $J_1^{(2n+1)}(t,x)$ in $(3.7)_2$, we arrive at (3.11). ∎

We turn to the calculation of $J_2^{(2n+1)}(t,x)$.

Lemma 3.2. *Let $n \geq 1$.*

(i) *For any $t \geq 1$ and $x \geq -n\sqrt{t}$,*

$$J_2^{(2n+1)}(t,x) = P\left(\frac{R^{(3)}_{n^2 t}-n^2 t}{\sqrt{n^2 t}} \geq x\right).$$

(ii) *There exists a constant $c > 0$ such that*

$$\sup_{x\geq -n\sqrt{t}} |J_2^{(2n+1)}(t,x)-\Phi(x)| \leq \frac{c}{\sqrt{t}}, \qquad t \geq 1. \tag{3.16}$$

Proof. We first prove (i). Let $d = 2n+1$. For $t \geq 1$ and $x \geq -n\sqrt{t}$, we then have $T = x\sqrt{t}+nt$. By (2.3) and (A.9), it follows that

$$\left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-1}\sinh^{2n-1} r\right) q_3(t,r)\sinh r \tag{3.17}$$
$$= \frac{(2n-1)!!}{n}\sinh(nr)\,\frac{e^{-t/2}}{(2\pi t)^{3/2}}\,\frac{r}{\sinh r}e^{-r^2/(2t)}\sinh r$$
$$= \frac{(2n-1)!!}{n}\,\frac{e^{-t/2}}{(2\pi t)^{3/2}} r e^{-r^2/(2t)}\sinh(nr)$$

and thus

$$\frac{e^{-(n^2-1)t/2}}{(2\pi)^{n-1}}\int_T^\infty \left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-1}\sinh^{2n-1} r\right) q_3(t,r)\sinh r\,\mathrm{d}r$$
$$= \frac{e^{-(n^2-1)t/2}}{(2\pi)^{n-1}}\,\frac{(2n-1)!!}{n}\,\frac{e^{-t/2}}{(2\pi t)^{3/2}}\int_T^\infty r e^{-r^2/(2t)}\sinh(nr)\,\mathrm{d}r$$
$$= \frac{1}{(2\pi)^{n-1}(2\pi t)^{3/2}}\,\frac{(2n-1)!!}{n}\int_T^\infty r e^{-n^2t/2} e^{-r^2/(2t)}\sinh(nr)\,\mathrm{d}r.$$

We also note that

$$\omega_{2n+1} = \frac{(2\pi)^{n-1}4\pi}{(2n-1)!!} = \frac{(2\pi)^{n-1}\omega_3}{(2n-1)!!}.$$

Then by (3.9) and (2.3),

$$J_2^{(2n+1)}(t,x) = \frac{\omega_{2n+1}}{(2\pi)^{n-1}(2\pi t)^{3/2}}\,\frac{(2n-1)!!}{n}\int_T^\infty r e^{-n^2t/2}e^{-r^2/(2t)}\sinh(nr)\,\mathrm{d}r$$
$$= \frac{\omega_3}{n(2\pi t)^{3/2}}\int_T^\infty r e^{-n^2t/2}e^{-r^2/(2t)}\sinh(nr)\,\mathrm{d}r$$
$$= \frac{\omega_3}{(2\pi n^2 t)^{3/2}}\int_{x\sqrt{n^2t}+n^2t}^\infty \frac{u}{\sinh u} e^{-n^2t/2}e^{-u^2/(2n^2t)}\sinh^2 u\,\mathrm{d}u$$
$$= \omega_3\int_{x\sqrt{n^2t}+n^2t}^\infty q_3(n^2t,u)\sinh^2 u\,\mathrm{d}u$$
$$= P\left(\frac{R^{(3)}_{n^2t}-n^2t}{\sqrt{n^2t}}\geq x\right).$$

At the second equality above, we have used the change of variables $u = nr$.

We next prove (ii). By (i),

$$J_2^{(2n+1)}(t,x) - \Phi(x) = P\left(\frac{R^{(3)}_{n^2t}-n^2t}{\sqrt{n^2t}}\geq x\right) - \Phi(x). \tag{3.18}$$

Since (1.4) is already proved for $d = 3$ in Section 3.1, the proof is complete. ■

We are now in a position to complete the proof of (1.4). By (3.10), (3.11) and (3.16), we have

$$\sup_{x\geq -n\sqrt{t}}\left|P\left(\frac{R_t^{(2n+1)}-nt}{\sqrt{t}}\geq x\right)-\Phi(x)\right|\lesssim\frac{1}{\sqrt{t}},\qquad t\geq 1.$$

The proof of (1.4) is complete for general odd dimensions.

3.3. Proof of (1.5) for odd dimensions. We start from (3.10) with $x=0$. Since $T=T(t,0)=nt$, we see by (3.8) that

$$J_1^{(2n+1)}(t,0)$$
$$=\omega_{2n+1}\sum_{m=1}^{n-1}\frac{e^{-(2n-m)mt/2}}{(2\pi)^m}q_{2n+1-2m}(t,nt)\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m-1}\sinh^{2n-1}r\bigg|_{r=nt}.$$

By (3.12) with $d=2n+1$ and $T=nt$, we have

$$e^{-(2n-m)mt/2}q_{2n+1-2m}(t,nt)$$
$$\asymp\frac{1}{t^{(2n+1-2m)/2}}\times e^{-(2n-m)mt/2}\exp\left(-\frac{(n-m)^2}{2}t-(n-m)nt-\frac{n^2t}{2}\right)$$
$$\times(1+nt+t)^{n-m-1}(1+nt).$$

Then by (3.13) with $d=2n+1$ and $T=nt$,

$$e^{-(2n-m)mt/2}\exp\left(-\frac{(n-m)^2}{2}t-(n-m)nt-\frac{n^2t}{2}\right)=e^{-(2n-m)nt}.$$

For any $t\geq 1$, since $(1+nt+t)^{n-m-1}(1+nt)\asymp t^{n-m}$, we get

$$e^{-(2n-m)mt/2}q_{2n+1-2m}(t,nt)\asymp\frac{e^{-(2n-m)nt}}{\sqrt{t}}.$$

On the other hand, it follows by (A.7) that for any m $(1\leq m\leq n-1)$, there exists a constant $c_1>0$ such that

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m-1}\sinh^{2n-1}r\bigg|_{r=nt}\geq c_1e^{(2n-m)nt}.$$

Hence by the argument above,

$$J_1^{(2n+1)}(t,0)\gtrsim\frac{1}{\sqrt{t}},\qquad t\geq 1. \tag{3.19}$$

Since (1.5) is already proved for $d=3$, we see from (3.18) that

$$J_2^{(2n+1)}(t,0)-\Phi(0)=P\left(\frac{R_{n^2t}^{(3)}-n^2t}{\sqrt{n^2t}}\geq 0\right)-\Phi(0)\gtrsim\frac{1}{\sqrt{t}},\qquad t\geq 1.$$

Combining this with (3.10) (for $x = 0$) and (3.19), we have

$$P\left(\frac{R_t^{(2n+1)} - nt}{\sqrt{t}} \geq 0\right) - \Phi(0) \gtrsim \frac{1}{\sqrt{t}}, \qquad t \geq 1.$$

We have thus completed the proof of (1.5) for general odd dimensions.

4. Proof of Theorem 1.1 for even dimensions

4.1. Proof of (1.4) for even dimensions. Suppose that $d = 2n$ for some $n \geq 1$. Then for any $t \geq 1$ and $x \geq -(2n-1)\sqrt{t}/2$, we have $T = x\sqrt{t} + (2n-1)t/2$. We define $J_1^{(2n)}(t,x)$ and $J_2^{(2n)}(t,x)$ as $(3.7)_2$ and $(3.7)_4$, respectively, with $d = 2n$:

$$J_1^{(2n)}(t,x) = \omega_{2n} \sum_{m=1}^{n-1} \frac{e^{-(2n-1-m)mt/2}}{(2\pi)^m} q_{2n-2m}(t,T) \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m-1} \sinh^{2n-2} r \Big|_{r=T}$$

and

$$J_2^{(2n)}(t,x) = \omega_{2n} \frac{e^{-n(n-1)t/2}}{(2\pi)^{n-1}} \int_T^\infty \left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-1} \sinh^{2n-2} r\right) q_2(t,r) \sinh r \, \mathrm{d}r,$$

where $J_1^{(2)}(t,x) = 0$ by the convention $\sum_{m=1}^0 = 0$. In what follows, we simply write $J_1(t,x) = J_1^{(2n)}(t,x)$ and $J_2(t,x) = J_2^{(2n)}(t,x)$. We have

$$P\left(\frac{R_t^{(2n)} - (2n-1)t/2}{\sqrt{t}} \geq x\right) - \Phi(x) = J_1(t,x) + (J_2(t,x) - \Phi(x)). \tag{4.1}$$

We now estimate the right hand side of (4.1). For an upper bound of $J_1(t,x)$, we can follow the proof of Lemma 3.1 to show

LEMMA 4.1. *There exists a constant $c > 0$ such that for any $t \geq 1$,*

$$\sup_{x \geq -(2n-1)\sqrt{t}/2} J_1(t,x) \leq \frac{c}{\sqrt{t}}. \tag{4.2}$$

For $J_2(t,x)$, we have

LEMMA 4.2. *There exists a constant $c > 0$ such that for any $t \geq 1$,*

$$\sup_{x \geq -(2n-1)\sqrt{t}/2} |J_2(t,x) - \Phi(x)| \leq \frac{c}{\sqrt{t}}. \tag{4.3}$$

Proof. Let

$$a_n(t) = \omega_{2n} \frac{e^{-n(n-1)t/2}}{(2\pi)^{n-1}} = \frac{\pi e^{-n(n-1)t/2}}{2^{n-2}(n-1)!}$$

and

$$K(t,x) = \int_T^\infty \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-1} \sinh^{2n-2} r \right) q_2(t,r) \sinh r \, \mathrm{d}r. \tag{4.4}$$

Then

$$J_2(t,x) = a_n(t) K(t,x). \tag{4.5}$$

By (2.2) and the Fubini theorem,

$$\begin{aligned} K(t,x) = {} & \frac{2^{1/2} e^{-t/8}}{(2\pi t)^{3/2}} \int_T^\infty s e^{-s^2/(2t)} \\ & \times \left\{ \int_T^s \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-1} \sinh^{2n-2} r \right) \frac{\sinh r}{(\cosh s - \cosh r)^{1/2}} \, \mathrm{d}r \right\} \mathrm{d}s. \end{aligned} \tag{4.6}$$

By integration by parts,

$$\begin{aligned} & \int_T^s \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-1} \sinh^{2n-2} r \right) \frac{\sinh r}{(\cosh s - \cosh r)^{1/2}} \, \mathrm{d}r \\ & = \left[-2 \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-1} \sinh^{2n-2} r \right) (\cosh s - \cosh r)^{1/2} \right]_{r=T}^{r=s} \\ & \quad + 2 \int_T^s \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n} \sinh^{2n-2} r \right) (\cosh s - \cosh r)^{1/2} \sinh r \, \mathrm{d}r \\ & = 2 \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-1} \sinh^{2n-2} r \right) \bigg|_{r=T} (\cosh s - \cosh T)^{1/2} \\ & \quad + 2 \int_T^s \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n} \sinh^{2n-2} r \right) (\cosh s - \cosh r)^{1/2} \sinh r \, \mathrm{d}r. \end{aligned}$$

Inductively, we get

$$\begin{aligned} & \int_T^s \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-1} \sinh^{2n-2} r \right) \frac{\sinh r}{(\cosh s - \cosh r)^{1/2}} \, \mathrm{d}r \\ = & \sum_{k=1}^{n-1} \frac{2^k}{(2k-1)!!} \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n-2+k} \right) \sinh^{2n-2} r \bigg|_{r=T} (\cosh s - \cosh T)^{k-1/2} \\ & + \frac{2^{n-1}}{(2n-3)!!} \int_T^s \left(\left(\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{2n-2} \sinh^{2n-2} r \right) \\ & \qquad \times (\cosh s - \cosh r)^{n-3/2} \sinh r \, \mathrm{d}r. \end{aligned} \tag{4.7}$$

Since (A.8) yields

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2n-2}\sinh^{2n-2} r = (2n-2)!,$$

we have

$$\begin{aligned}\int_T^s & \left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2n-2}\sinh^{2n-2} r\right)(\cosh s-\cosh r)^{n-3/2}\sinh r\,\mathrm{d}r \\ &= (2n-2)!\int_T^s (\cosh s-\cosh r)^{n-3/2}\sinh r\,\mathrm{d}r \\ &= \frac{2(2n-2)!}{2n-1}(\cosh s-\cosh T)^{n-1/2}\end{aligned}$$

and thus

$$\begin{aligned}\frac{2^{n-1}}{(2n-3)!!}\int_T^s & \left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2n-2}\sinh^{2n-2} r\right)(\cosh s-\cosh r)^{n-3/2}\sinh r\,\mathrm{d}r \\ &= \frac{2^n(2n-2)!}{(2n-1)!!}(\cosh s-\cosh T)^{n-1/2} \\ &= \frac{2^{2n-1}(n-1)!}{2n-1}(\cosh s-\cosh T)^{n-1/2}.\end{aligned}$$

Combining this with (4.6) and (4.7), we obtain

$$\begin{aligned}(4.8)\qquad K(t,x) & \\ &= \frac{2^{1/2}e^{-t/8}}{(2\pi t)^{3/2}}\sum_{k=1}^{n-1}\frac{2^k}{(2k-1)!!}\left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-2+k}\right)\sinh^{2n-2} r\Big|_{r=T} \\ &\qquad\times\int_T^\infty s e^{-s^2/(2t)}(\cosh s-\cosh T)^{k-1/2}\,\mathrm{d}s \\ &\quad+\frac{2^{2n-1}(n-1)!}{2n-1}\frac{2^{1/2}e^{-t/8}}{(2\pi t)^{3/2}}\int_T^\infty s e^{-s^2/(2t)}(\cosh s-\cosh T)^{n-1/2}\,\mathrm{d}s \\ &= K_1(t,x)+K_2(t,x),\end{aligned}$$

whence by (4.5),

$$\begin{aligned}(4.9)\qquad J_2(t,x)-\Phi(x) &= a_n(t)(K_1(t,x)+K_2(t,x))-\Phi(x) \\ &= a_n(t)K_1(t,x)+(a_n(t)K_2(t,x)-\Phi(x)).\end{aligned}$$

We first estimate $a_n(t)K_1(t,x)$. By (A.6), there exists a constant $c_1>0$, independent of t, T and k ($1\le k\le n-1$), such that

$$(4.10)\qquad 0 \le \left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-2+k}\right)\sinh^{2n-2} r\,\Big|_{r=T} \le c_1 e^{(n-k)T} = c_1 e^{(n-k)(x\sqrt{t}+(2n-1)t/2)}.$$

On the other hand, for any $s \ge T$,

$$(4.11)\qquad 0 \le (\cosh s - \cosh T)^{k-1/2} \le \cosh^{k-1/2} s \le e^{(k-1/2)s}.$$

Then by the change of variables $s = u\sqrt{t} + (2n-1)t/2$,

$$\begin{aligned}\int_T^\infty & se^{-s^2/(2t)}(\cosh s - \cosh T)^{k-1/2}\,\mathrm{d}s \\ &\le \int_T^\infty se^{-s^2/(2t)}e^{(k-1/2)s}\,\mathrm{d}s \\ &= \sqrt{t}\int_x^\infty \left(u\sqrt{t} + \frac{2n-1}{2}t\right)e^{-(u+(2n-1)\sqrt{t}/2)^2/2}e^{(k-1/2)(u\sqrt{t}+(2n-1)t/2)}\,\mathrm{d}u \\ &= t\sqrt{t}e^{t/8}e^{-(n^2-2nk+k)t/2}\int_x^\infty \left(\frac{u}{\sqrt{t}} + \frac{2n-1}{2}\right)e^{-u^2/2}e^{-(n-k)u\sqrt{t}}\,\mathrm{d}u.\end{aligned}$$

Since

$$\begin{aligned}\int_x^\infty & \left(\frac{u}{\sqrt{t}} + \frac{2n-1}{2}\right)e^{-u^2/2}e^{-(n-k)u\sqrt{t}}\,\mathrm{d}u \\ &\le c_2\int_x^\infty e^{-(n-k)u\sqrt{t}}\,\mathrm{d}u = \frac{c_2e^{-(n-k)x\sqrt{t}}}{(n-k)\sqrt{t}} \le \frac{c_2e^{-(n-k)x\sqrt{t}}}{\sqrt{t}},\end{aligned}$$

we obtain

$$\int_T^\infty se^{-s^2/(2t)}(\cosh s - \cosh T)^{k-1/2}\,\mathrm{d}s \le c_2te^{t/8}e^{-(n^2-2nk+k)t/2}e^{-(n-k)x\sqrt{t}},$$

which yields

$$\frac{2^{1/2}e^{-t/8}}{(2\pi t)^{3/2}}\int_T^\infty se^{-s^2/(2t)}(\cosh s - \cosh T)^{k-1/2}\,\mathrm{d}s \le c_3e^{-(n^2-2nk+k)t/2}\frac{e^{-(n-k)x\sqrt{t}}}{\sqrt{t}}.$$

Combining this with (4.10), we get

$$0 \le \left(\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n-2+k}\right)\sinh^{2n-2} r\,\bigg|_{r=T}\frac{2^{1/2}e^{-t/8}}{(2\pi t)^{3/2}}$$
$$\times \int_T^\infty s e^{-s^2/(2t)}(\cosh s - \cosh T)^{k-1/2}\,\mathrm{d}s$$
$$\le c_4 e^{(n-k)(x\sqrt{t}+(n-1/2)t)} e^{-(n^2-2nk+k)t/2}\frac{e^{-(n-k)x\sqrt{t}}}{\sqrt{t}} = \frac{c_4 e^{n(n-1)t/2}}{\sqrt{t}}.$$

This implies that

$$0 \le K_1(t,x) \le \frac{c_5 e^{n(n-1)t/2}}{\sqrt{t}}$$

and thus

$$a_n(t)K_1(t,x) = \frac{\pi e^{-n(n-1)t/2}}{2^{n-2}(n-1)!}K_1(t,x) \le \frac{c_6}{\sqrt{t}}. \tag{4.12}$$

Let us next estimate $|a_n(t)K_2(t,x) - \Phi(x)|$. By definition,

(4.13)
$$a_n(t)K_2(t,x) = b_n(t)\int_T^\infty s e^{-s^2/(2t)}(\cosh s - \cosh T)^{n-1/2}\,\mathrm{d}s$$
$$= b_n(t)\int_T^\infty \left(s - \frac{2n-1}{2}t\right)e^{-s^2/(2t)}(\cosh s - \cosh T)^{(2n-1)/2}\,\mathrm{d}s$$
$$+ \frac{2n-1}{2}t b_n(t)\int_T^\infty e^{-s^2/(2t)}(\cosh s - \cosh T)^{(2n-1)/2}\,\mathrm{d}s,$$

where

$$b_n(t) = \frac{2^n}{(2n-1)\sqrt{\pi}}\,\frac{e^{-(2n-1)^2 t/8}}{t^{3/2}}. \tag{4.14}$$

By the change of variables $y = s/\sqrt{t} - (2n-1)\sqrt{t}/2$ we also have

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-y^2/2}\,\mathrm{d}y \tag{4.15}$$
$$= \frac{2n-1}{2}t b_n(t)\int_T^\infty e^{-s^2/(2t)}\left(\frac{e^s}{2}\right)^{(2n-1)/2}\mathrm{d}s.$$

Then

$$
\begin{aligned}
&a_n(t)K_2(t,x) - \Phi(x)\\
&= b_n(t)\int_T^\infty \Big(s - \frac{2n-1}{2}t\Big) e^{-s^2/(2t)} (\cosh s - \cosh T)^{n-1/2}\,\mathrm{d}s\\
&\quad - \frac{2n-1}{2} t b_n(t) \int_T^\infty e^{-s^2/(2t)} \bigg\{ \Big(\frac{e^s}{2}\Big)^{(2n-1)/2} - (\cosh s - \cosh T)^{(2n-1)/2} \bigg\}\,\mathrm{d}s.
\end{aligned}
$$

For $s \ge T$, since $e^s/2 \ge \cosh s - \cosh T$, we have

$$
(e^s/2)^{(2n-1)/2} \ge (\cosh s - \cosh T)^{(2n-1)/2}.
$$

Therefore,

$$
\begin{aligned}
&|a_n(t)K_2(t,x) - \Phi(x)| \qquad (4.16)\\
&\le b_n(t)\int_T^\infty \Big|s - \frac{2n-1}{2}t\Big| e^{-s^2/(2t)} (\cosh s - \cosh T)^{n-1/2}\,\mathrm{d}s\\
&\quad + \frac{2n-1}{2} t b_n(t) \int_T^\infty e^{-s^2/(2t)} \bigg\{ \Big(\frac{e^s}{2}\Big)^{(2n-1)/2} - (\cosh s - \cosh T)^{(2n-1)/2} \bigg\}\,\mathrm{d}s\\
&= b_n(t)L_1(t,x) + \frac{2n-1}{2} t b_n(t) L_2(t,x).
\end{aligned}
$$

By (4.11) and the change of variables $s = u\sqrt{t} + (2n-1)t/2$,

$$
\begin{aligned}
L_1(t,x) &\le \int_T^\infty \Big|s - \frac{2n-1}{2}t\Big| e^{-s^2/(2t)} e^{(2n-1)s/2}\,\mathrm{d}s\\
&= e^{(2n-1)^2 t/8} t \int_x^\infty |u| e^{-u^2/2}\,\mathrm{d}u\\
&\le e^{(2n-1)^2 t/8} t \int_{-\infty}^\infty |u| e^{-u^2/2}\,\mathrm{d}u = 2e^{(2n-1)^2 t/8} t.
\end{aligned}
$$

Hence, by (4.14),

$$
b_n(t)L_1(t,x) \le \frac{2^{n+1}}{(2n-1)\sqrt{\pi}} \frac{1}{\sqrt{t}}. \tag{4.17}
$$

On the other hand, for any $s \ge T$,

$$
\begin{aligned}
(\cosh s - \cosh T)^{(2n-1)/2} &= \cosh^{(2n-1)/2} s \Big(1 - \frac{\cosh T}{\cosh s}\Big)^{(2n-1)/2}\\
&= \Big(\frac{e^s}{2}\Big)^{(2n-1)/2} (1 + e^{-2s})^{(2n-1)/2} \Big(1 - \frac{\cosh T}{\cosh s}\Big)^{(2n-1)/2}
\end{aligned}
$$

and so

$$\begin{aligned}\left(\frac{e^s}{2}\right)^{(2n-1)/2} &- (\cosh s - \cosh T)^{(2n-1)/2} \\ &= \left(\frac{e^s}{2}\right)^{(2n-1)/2}\left(1 - (1+e^{-2s})^{(2n-1)/2}\left(1 - \frac{\cosh T}{\cosh s}\right)^{(2n-1)/2}\right) \\ &\leq \left(\frac{e^s}{2}\right)^{(2n-1)/2}\left(1 - \left(1 - \frac{\cosh T}{\cosh s}\right)^{(2n-1)/2}\right).\end{aligned}$$

Note that for any $p \geq 0$,

$$0 \leq 1 - (1-v)^p \leq pv, \qquad 0 \leq v \leq 1. \tag{4.18}$$

Then for any $s \geq T$,

$$0 \leq 1 - \left(1 - \frac{\cosh T}{\cosh s}\right)^{(2n-1)/2} \leq \frac{2n-1}{2}\,\frac{\cosh T}{\cosh s} \leq (2n-1)e^{-(s-T)}.$$

Hence

$$\left(\frac{e^s}{2}\right)^{(2n-1)/2} - (\cosh s - \cosh T)^{(2n-1)/2} \leq (2n-1)e^{(2n-1)s/2}e^{-(s-T)},$$

which yields

$$\begin{aligned}L_2(t,x) &= \int_T^\infty e^{-s^2/(2t)}\left\{\left(\frac{e^s}{2}\right)^{(2n-1)/2} - (\cosh s - \cosh T)^{(2n-1)/2}\right\} \mathrm{d}s \\ &\leq (2n-1)\int_T^\infty e^{-s^2/(2t)}e^{(2n-1)s/2}e^{-(s-T)}\,\mathrm{d}s \\ &= (2n-1)e^{(2n-1)^2t/8}\int_T^\infty e^{-(s-(2n-1)t/2)^2/(2t)}e^{-(s-T)}\,\mathrm{d}s \\ &\leq (2n-1)e^{(2n-1)^2t/8}\int_T^\infty e^{-(s-T)}\,\mathrm{d}s = (2n-1)e^{(2n-1)^2t/8}.\end{aligned}$$

Therefore, by (4.14),

$$\frac{2n-1}{2}tb_n(t)L_2(t,x) \leq \frac{2^{n-1}(2n-1)}{\sqrt{\pi}}\,\frac{1}{\sqrt{t}}.$$

Combining this with (4.16) and (4.17), for any $t \geq 1$ we obtain

$$\sup_{x \geq -(2n-1)t/2} |a_n(t)K_2(t,x) - \Phi(x)| \leq \frac{2^{n-1}}{\sqrt{\pi}}\left(2n-1+\frac{4}{2n-1}\right)\frac{1}{\sqrt{t}}. \tag{4.19}$$

This together with (4.9) and (4.12) leads to (4.3). ∎

By (4.2) and (4.3), we have completed the proof of (1.4) for general even dimensions.

4.2. Proof of (1.5) for $d = 2$. We now prove that there exists a constant $c > 0$ such that

$$P\left(\frac{R_t^{(2)} - t/2}{\sqrt{t}} \geq 0\right) - \Phi(0) \geq \frac{c}{\sqrt{t}}, \qquad t \geq 1.$$

Let $R_t = R_t^{(2)}$. Then by (2.1) with $\omega_2 = 2$,

$$P\left(\frac{R_t - t/2}{\sqrt{t}} \geq 0\right) = P\left(R_t \geq \frac{t}{2}\right) = 2\int_{t/2}^{\infty} q_2(t,r)\sinh r\,\mathrm{d}r.$$

On the other hand, by (4.4) and (4.8) with $n = 1$ and $x = 0$,

$$2\int_{t/2}^{\infty} q_2(t,r)\sinh r\,\mathrm{d}r = \frac{2e^{-t/8}}{t\sqrt{\pi t}}\int_{t/2}^{\infty} re^{-r^2/(2t)}(\cosh r - \cosh(t/2))^{1/2}\,\mathrm{d}r.$$

Moreover, by integration by parts,

$$\frac{2e^{-t/8}}{t\sqrt{\pi t}}\int_{t/2}^{\infty} re^{-r^2/(2t)}(\cosh r - \cosh(t/2))^{1/2}\,\mathrm{d}r = \frac{e^{-t/8}}{\sqrt{\pi t}}\int_{t/2}^{\infty} e^{-r^2/(2t)}\frac{\sinh r}{(\cosh r - \cosh(t/2))^{1/2}}\,\mathrm{d}r.$$

Therefore,

$$P\left(\frac{R_t - t/2}{\sqrt{t}} \geq 0\right) = \frac{e^{-t/8}}{\sqrt{2\pi t}}\int_{t/2}^{\infty} e^{-r^2/(2t)}\frac{\sqrt{2}\sinh r}{(\cosh r - \cosh(t/2))^{1/2}}\,\mathrm{d}r.$$

On the other hand, by (4.15) with $x = 0$ and $n = 1$,

$$\Phi(0) = \frac{e^{-t/8}}{\sqrt{2\pi t}}\int_{t/2}^{\infty} e^{-r^2/(2t)}e^{r/2}\,\mathrm{d}r.$$

We thus have

$$\begin{aligned}(4.20)\qquad & P\left(\frac{R_t - t/2}{\sqrt{t}} \geq 0\right) - \Phi(0) \\ &= \frac{e^{-t/8}}{\sqrt{2\pi t}}\int_{t/2}^{\infty} e^{-r^2/(2t)}e^{r/2}\left(\frac{\sqrt{2}\,e^{-r/2}\sinh r}{(\cosh r - \cosh(t/2))^{1/2}} - 1\right)\mathrm{d}r \\ &= \frac{e^{-t/8}}{\sqrt{2\pi t}}\int_{t/2}^{\infty} e^{-r^2/(2t)}e^{r/2}F_t(r)\,\mathrm{d}r\end{aligned}$$

with

$$F_t(r) = \frac{\sqrt{2}\,e^{-r/2}\sinh r}{(\cosh r - \cosh(t/2))^{1/2}} - 1.$$

Then $F_t(r) \geq 0$ for any $t \geq 1$ and $r \geq t/2$ because

$$(\sqrt{2}\,e^{-r/2}\sinh r)^2 - (\cosh r - \cosh(t/2)) = \frac{e^{-3r} - 3e^{-r}}{2} + \frac{e^{t/2} + e^{-t/2}}{2} \geq -\frac{3e^{-t/2}}{2} + \frac{e^{t/2} + e^{-t/2}}{2} = \frac{e^{t/2} - 2e^{-t/2}}{2} \geq 0. \tag{4.21}$$

Hence the right hand side of (4.20) is positive.

For $t \geq 1$ and $r \geq t/2$, let

$$F_t(r) = \frac{\sqrt{2}\,e^{-r/2}\sinh r}{\cosh^{1/2} r}\left(1 - \frac{\cosh(t/2)}{\cosh r}\right)^{-1/2} - 1 \tag{4.22}$$

$$= \left(1 - \frac{\cosh(t/2)}{\cosh r}\right)^{-1/2} - 1 - \left(1 - \frac{\cosh(t/2)}{\cosh r}\right)^{-1/2}\left(1 - \frac{\sqrt{2}\,e^{-r/2}\sinh r}{\cosh^{1/2} r}\right).$$

Since

$$\frac{1}{\sqrt{1-v}} - 1 \geq 1 - \sqrt{1-v} \geq \frac{1}{2}v, \qquad 0 \leq v < 1,$$

we obtain

$$\left(1 - \frac{\cosh(t/2)}{\cosh r}\right)^{-1/2} - 1 \geq \frac{1}{2}\,\frac{\cosh(t/2)}{\cosh r} \geq \frac{1}{4}e^{-(r-t/2)}. \tag{4.23}$$

We also note that

$$0 \leq 1 - \frac{1-v}{\sqrt{1+v}} = \frac{3v - v^2}{\sqrt{1+v}(\sqrt{1+v} + 1 - v)} \leq 3v, \qquad 0 \leq v \leq 1,$$

and so

$$0 \leq 1 - \frac{\sqrt{2}\,e^{-r/2}\sinh r}{\cosh^{1/2} r} = 1 - \frac{e^{r/2} - e^{-3r/2}}{\sqrt{e^r + e^{-r}}} = 1 - \frac{1 - e^{-2r}}{\sqrt{1 + e^{-2r}}} \leq 3e^{-2r}. \tag{4.24}$$

If we assume in addition that $r \geq 1 + t/2$, then

$$1 - \frac{\cosh(t/2)}{\cosh r} \geq 1 - \frac{\cosh(t/2)}{\cosh(1 + t/2)} \geq 1 - \frac{1}{e} > 0.$$

Combining this with (4.24) we have, for any $r \geq 1 + t/2$,

$$\left(1 - \frac{\cosh(t/2)}{\cosh r}\right)^{-1/2}\left(1 - \frac{\sqrt{2}\,e^{-r/2}\sinh r}{\cosh^{1/2} r}\right) \leq 3(1 - e^{-1})^{-1/2}e^{-2r}. \tag{4.25}$$

Therefore, by (4.22), (4.23) and (4.25), for any $r \geq 1 + t/2$ we have

$$F_t(r) \geq \frac{1}{4}e^{-(r-t/2)} - 3(1 - e^{-1})^{-1/2}e^{-2r}. \tag{4.26}$$

By (4.26), we get

$$\int_{t/2}^{\infty} e^{-r^2/(2t)} e^{r/2} F_t(r)\,\mathrm{d}u \ge \int_{1+t/2}^{\infty} e^{-r^2/(2t)} e^{r/2} F_t(r)\,\mathrm{d}r \ge \frac{1}{4} e^{t/2} \int_{1+t/2}^{\infty} e^{-r^2/(2t)-r/2}\,\mathrm{d}r - 3(1-e^{-1})^{-1/2} \int_{1+t/2}^{\infty} e^{-r^2/(2t)-3r/2}\,\mathrm{d}r. \tag{4.27}$$

Note that for any $c > 0$, by (2.8) we have

$$\begin{aligned}\int_{1+t/2}^{\infty} & e^{-r^2/(2t)} e^{-cr}\,\mathrm{d}r \\ &= e^{c^2t/2} \int_{1+t/2}^{\infty} e^{-(r+ct)^2/(2t)}\,\mathrm{d}r = e^{c^2t/2}\sqrt{t} \int_{(c+1/2)\sqrt{t}+1/\sqrt{t}}^{\infty} e^{-u^2/2}\,\mathrm{d}u \\ &\asymp e^{c^2t/2} e^{-(c+1/2)^2t/2} = e^{-(4c+1)t/8}.\end{aligned} \tag{4.28}$$

At the second equality above, we have used the change of variables $u = (r+ct)/\sqrt{t}$. Hence by (4.27), and taking $c = 1/2$ and $c = 3/2$ in (4.28), for all large $t \ge 1$ we get

$$\int_{t/2}^{\infty} e^{-r^2/(2t)} e^{r/2} F_t(r)\,\mathrm{d}u \ge c_1 e^{t/2} \cdot e^{-3t/8} - c_2 e^{-7t/8} \ge c_3 e^{t/8}. \tag{4.29}$$

Recall that the integral above is positive for any $t \ge 1$. By (4.20) and (4.29) we then have

$$P\Big(\frac{R_t - t/2}{\sqrt{t}} \ge 0\Big) - \Phi(0) \gtrsim \frac{1}{\sqrt{t}}, \qquad t \ge 1.$$

Thus, we have arrived at the desired conclusion.

REMARK 4.3. At the moment, it is unknown whether (1.5) is valid for even $d \ge 4$. To explain this, let us apply the argument for $d = 2$ to general even dimension $d = 2n$ with $n \ge 2$. Fix $x \ge 0$. Then by (4.1) and (4.9),

$$\begin{aligned}P\Big(\frac{R_t^{(2n)} - (2n-1)t/2}{\sqrt{t}} \ge x\Big) - \Phi(x) &= J_1(t,x) + (J_2(t,x) - \Phi(x)) \\ &= J_1(t,x) + a_n(t)K_1(t,x) + (a_n(t)K_2(t,x) - \Phi(x)).\end{aligned} \tag{4.30}$$

By the expression in the first line of (4.13), and integration by parts,

$$a_n(t)K_2(t,x) = b_n(t)\int_T^\infty se^{-s^2/(2t)}(\cosh s-\cosh T)^{n-1/2}\,\mathrm{d}s$$
$$= \frac{2n-1}{2}tb_n(t)\int_T^\infty e^{-s^2/(2t)}\sinh s\,(\cosh s-\cosh T)^{n-3/2}\,\mathrm{d}s.$$

Hence if we let

$$G_t(s) = 1-\left(\frac{2}{e^s}\right)^{(2n-1)/2}\sinh s\,(\cosh s-\cosh T)^{n-3/2}$$
$$= 1-(1-e^{-2s})(1+e^{-2s}-2e^{-s}\cosh T)^{n-3/2}, \qquad s\geq T, \tag{4.31}$$

then by (4.15),

$$a_n(t)K_2(t,x)-\Phi(x) = -\frac{2n-1}{2}tb_n(t)\int_T^\infty e^{-s^2/(2t)}\left(\frac{e^s}{2}\right)^{(2n-1)/2}G_t(s)\,\mathrm{d}s.$$

If $n=1$, then $G_t(s)$ is negative by a similar calculation to (4.21) and so the right hand side above is positive. On the other hand, if $n\geq 2$, then we can see by (4.31) that the function $G_t(s)$ is positive and decreasing on $[T,\infty)$, and such that $G_t(T)=1$ and $G_t(s)\to 0$ as $s\to\infty$. Therefore, (4.19) implies that for any $t\geq 1$,

$$a_n(t)K_2(t,x)-\Phi(x) \geq -|a_n(t)K_2(t,x)-\Phi(x)|$$
$$\geq -\frac{2^{n-1}}{\sqrt{\pi}}\left(2n-1+\frac{4}{2n-1}\right)\frac{1}{\sqrt{t}} = -\frac{c_*}{\sqrt{t}}.$$

We also see that, by an argument similar to that for (3.19) and (4.12) together with (A.7), there exist positive constants c_1 and c_2 such that for any $t\geq 1$,

$$J_1(t,x)\geq \frac{c_1}{\sqrt{t}}, \qquad a_n(t)K_1(t,x)\geq \frac{c_2}{\sqrt{t}}.$$

Hence (4.30) yields, for any $t\geq 1$,

$$P\left(\frac{R_t^{(2n)}-(2n-1)t/2}{\sqrt{t}}\geq x\right)-\Phi(x)\geq \frac{c_1+c_2-c_*}{\sqrt{t}}.$$

However, since we could not get effective estimates for c_1 and c_2, it is unknown whether the right hand side above is positive or not. We note that the constants c_1 and c_2 come from (2.5) and (A.7), respectively.

A. Appendix. To prove Theorem 1.1, we establish the following two lemmas.

Lemma A.1. *Let $n \geq 1$ and $r \neq 0$.*

(i) *For any $k = 1, 2, \ldots,$*

$$\text{(A.1)} \quad \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k} \sinh^{2n-1} r = \sum_{l=0}^{k} \frac{(2k)!}{2^{k-l}(k-l)!(2l)!}\Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big) \times \cosh^{2l} r \sinh^{2n-1-(2k+2l)} r.$$

For any $k = 0, 1, \ldots,$

$$\text{(A.2)} \quad \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k+1} \sinh^{2n-1} r = \sum_{l=1}^{k+1} \frac{(2k+1)!}{2^{k+1-l}(k+1-l)!(2l-1)!} \times \Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big) \cosh^{2l-1} r \sinh^{2n-1-(2k+2l)} r.$$

(ii) *For any $k = 1, \ldots, n,$*

$$\text{(A.3)} \quad \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k} \sinh^{2n} r = \sum_{l=0}^{k\wedge(n-k)} \frac{(2k)!}{2^{k-l}(k-l)!(2l)!}\Big(\prod_{m=0}^{k+l-1}(2n-2m)\Big) \cosh^{2l} r \sinh^{2n-(2k+2l)} r.$$

For any $k = 0, 1, \ldots, n-1,$

$$\text{(A.4)} \quad \left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k+1} \sinh^{2n} r = \sum_{l=1}^{(k+1)\wedge(n-k)} \frac{(2k+1)!}{2^{k+1-l}(k+1-l)!(2l-1)!} \times \Big(\prod_{m=0}^{k+l-1}(2n-2m)\Big) \cosh^{2l-1} r \sinh^{2n-(2k+2l)} r.$$

Proof. Fix $n \geq 1$. Since the proof of (i) is similar to that of (ii), we present the former only.

We prove (A.1) by induction. For $k = 1$, we have

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2} \sinh^{2n-1} r = (2n-1)\sinh^{2n-3} r + (2n-1)(2n-3)\cosh^2 r \sinh^{2n-5} r,$$

whence (A.1) is valid for $k=1$. Suppose that it holds for some $k \geq 1$. Since

$$\begin{aligned}\frac{\partial}{\partial r}&(\cosh^{2l} r \sinh^{2n-1-(2k+2l)} r)\\ &= 2l \cosh^{2l-1} r \sinh^{2n-(2k+2l)} r\\ &\quad + (2n-1-(2k+2l)) \cosh^{2l+1} r \sinh^{2n-2-(2k+2l)} r,\end{aligned}$$

we obtain

$$\tag{A.5} \frac{\partial}{\partial r}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k} \sinh^{2n-1} r$$
$$\begin{aligned}
&= \sum_{l=0}^{k} \frac{(2k)!}{2^{k-l}(k-l)!(2l)!}\Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big)\frac{\partial}{\partial r}(\cosh^{2l} r \sinh^{2n-1-(2k+2l)} r)\\
&= \sum_{l=1}^{k} \frac{(2k)!}{2^{k-l}(k-l)!(2l-1)!}\Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big)\cosh^{2l-1} r \sinh^{2n-(2k+2l)} r\\
&\quad + \sum_{l=0}^{k} \frac{(2k)!}{2^{k-l}(k-l)!(2l)!}\Big(\prod_{m=0}^{k+l}(2n-1-2m)\Big)\cosh^{2l+1} r \sinh^{2n-2-(2k+2l)} r\\
&= \sum_{l=1}^{k} \frac{(2k)!}{2^{k-l}(k-l)!(2l-1)!}\Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big)\cosh^{2l-1} r \sinh^{2n-(2k+2l)} r\\
&\quad + \sum_{l=1}^{k+1} \frac{(2k)!}{2^{k-l+1}(k-l+1)!(2l-2)!}\\
&\quad \times \Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big)\cosh^{2l-1} r \sinh^{2n-(2k+2l)} r\\
&= \sum_{l=1}^{k+1} \frac{(2k+1)!}{2^{k+1-l}(k+1-l)!(2l-1)!}\Big(\prod_{m=0}^{k+l-1}(2n-1-2m)\Big)\\
&\quad \times \cosh^{2l-1} r \sinh^{2n-(2k+2l)} r.
\end{aligned}$$

Therefore, (A.2) holds. Using it we obtain, in a similar way to (A.5),

$$\begin{aligned}
&\frac{\partial}{\partial r}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k+1} \sinh^{2n-1} r\\
&= \sum_{l=0}^{k+1} \frac{(2k+2)!}{2^{k+1-l}(k+1-l)!(2l)!}\Big(\prod_{m=0}^{k+l}(2n-1-2m)\Big)\cosh^{2l} r \sinh^{2n-2-(2k+2l)} r
\end{aligned}$$

and thus

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k+2}\sinh^{2n-1} r$$
$$= \sum_{l=0}^{k+1}\frac{(2k+2)!}{2^{k+1-l}(k+1-l)!(2l)!}\Big(\prod_{m=0}^{k+l}(2n-1-2m)\Big)\cosh^{2l} r \sinh^{2n-3-(2k+2l)} r.$$

That is, (A.1) is valid for k replaced with $k+1$. Hence the induction is complete. In particular, our argument above also implies (A.2) for $k \geq 1$. Since we can verify (A.2) for $k = 0$ by direct computation, the proof is complete. ■

REMARK A.2. Let m and n be positive integers with $n \geq 2m$. Then by Lemma A.1, we have the following two clauses:

- There exists a constant $c_1 > 0$ such that

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m}\sinh^{n} r \leq c_1 e^{(n-m)r}, \qquad r > 0. \tag{A.6}$$

  Note that for even n we see from (A.3) and (A.4) that (A.6) is still valid for $n \geq m$.
- For any $r_0 > 0$, there exists a constant $c_2 > 0$ such that

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{m}\sinh^{n} r \geq c_2 e^{(n-m)r}, \qquad r \geq r_0. \tag{A.7}$$

LEMMA A.3. *Let $r \neq 0$.*

(i) *For $n \geq 1$,*

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2n}\sinh^{2n} r = (2n)!. \tag{A.8}$$

(ii) *For $n \geq 1$,*

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{n}\sinh^{2n+1} r = \frac{(2n+1)!!}{n+1}\sinh((n+1)r). \tag{A.9}$$

Proof. (i) follows from (A.3). To show (ii), we calculate the left hand side of (A.9) for odd n and even n separately. Namely, we will prove that for any $k \geq 1$,

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k-1}\sinh^{4k-1} r = \frac{(4k-1)!!}{2k}\sinh(2kr) \tag{A.10}$$

and

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k}\sinh^{4k+1} r = \frac{(4k+1)!!}{2k+1}\sinh((2k+1)r). \tag{A.11}$$

By (A.2),

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k-1}\sinh^{4k-1} r$$
$$= \sum_{l=1}^{k}\frac{(2k-1)!}{2^{k-l}(k-l)!(2l-1)!}\Big(\prod_{m=0}^{k+l-2}(4k-1-2m)\Big)\cosh^{2l-1} r\sinh^{2k-2l+1} r$$
$$= \sum_{j=0}^{k-1}\frac{(2k-1)!}{2^{j}j!(2k-2j-1)!}\Big(\prod_{m=0}^{2k-j-2}(4k-1-2m)\Big)\cosh^{2k-(2j+1)} r\sinh^{2j+1} r,$$

where we set $j=k-l$ in the second equality above. If $0\le j\le k-1$, then

$$\frac{(2k-1)!}{2^{j}j!(2k-2j-1)!}\prod_{m=0}^{2k-j-2}(4k-1-2m) = \frac{(2k-1)!}{(2j)!!(2k-2j-1)!}\prod_{l=j+2}^{2k}(2l-1)$$
$$= \frac{(4k-1)!!}{2k}\frac{(2k)!}{(2j+1)!(2k-(2j+1))!} = \frac{(4k-1)!!}{2k}\binom{2k}{2j+1}$$

and thus

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k-1}\sinh^{4k-1} r$$
$$= \frac{(4k-1)!!}{2k}\sum_{j=0}^{k-1}\binom{2k}{2j+1}\cosh^{2k-(2j+1)} r\sinh^{2j+1} r = \frac{(4k-1)!!}{2k}\sinh 2kr.$$

The proof of (A.10) is complete.

By (A.1) and a similar argument to the above,

$$\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^{2k}\sinh^{4k+1} r$$
$$= \sum_{l=0}^{k}\frac{(2k)!}{2^{k-l}(k-l)!(2l)!}\Big(\prod_{m=0}^{k+l-1}(4k+1-2m)\Big)\cosh^{2l} r\sinh^{2k-(2l-1)} r$$
$$= \sum_{j=0}^{k}\frac{(2k)!}{2^{j}j!(2(k-j))!}\Big(\prod_{i=j+1}^{2k}(2i+1)\Big)\cosh^{2(k-j)} r\sinh^{2j+1} r$$
$$= \frac{(4k+1)!!}{2k+1}\sum_{j=0}^{k}\binom{2k+1}{2j+1}\cosh^{2(k-j)} r\sinh^{2j+1} r$$
$$= \frac{(4k+1)!!}{2k+1}\sinh((2k+1)r).$$

The proof of (A.11) is complete. ■

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