# Volume growth, big jump, and essential spectrum for regular Dirichlet forms

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## 1. Introduction

**Our objective in this talk:**

**Upper bound of the bottom of the essential spectrum in terms of the volume growth and big jump rate**

$E$: locally compact separable metric space

$m$: positive Radon measure on $E$ with full support

$(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2$ $(= L^2(E; m))$

$(L, D(L))$: $L^2$-generator of $(\mathcal{E}, \mathcal{F})$ $(\mathcal{E}(u, v) = (-Lu, v)_{L^2})$

$T_t = e^{-Lt}\ (t > 0)$: **Markovian semigroup on** $L^2$

$$(0 \le f \le 1 \Rightarrow 0 \le T_t f \le 1\ (t > 0))$$

**The Dirichlet form** $(\mathcal{E}, \mathcal{F})$ **on** $L^2(E; m)$ **is characterized by**

- **Beurling-Deny expression (with no killing part):**

$$\mathcal{E}(u, u) = \frac{1}{2}\mu^c_{\langle u \rangle}(E) + \iint_{E \times E \setminus \mathrm{diag}} (u(x) - u(y))^2\, J(\mathrm{d}x, \mathrm{d}y)$$

- **Underlying measure** $m$

**Question.** **How these factors determine the properties of Markovian semigroups?**

$T_t$ is compact on $L^2$

$\underset{\text{def}}{\iff}$ $\forall B \subset L^2$ (bdd.), $T_t B$ is relatively compact in $L^2$

$\sigma_{\mathrm{ess}}(-L)$: essential spectrum of $-L$

$T_t$ is compact for some/any $t > 0$

$\iff$ $\sigma_{\mathrm{ess}}(-L) = \emptyset$, i.e., the spectrum of $-L$ is discrete

$\iff$ $\inf \sigma_{\mathrm{ess}}(-L) = \infty$

$\lambda_{\mathrm{e}} := \inf \sigma_{\mathrm{ess}}(-L)$: bottom of the essential spectrum

$\rightsquigarrow$ characteristic quantity of noncompactness

## Brooks (1981/1984): Volume growth and upper bound of $\lambda_{\mathrm{e}}$

$(M, g)$: complete and noncompact Riemannian manifold

$L = \Delta$: Laplace-Beltrami operator on $M$

$$\rightsquigarrow \mathcal{E}(u, u) = \int_M \langle \nabla u, \nabla u \rangle_g \, \mathrm{d}v = \frac{1}{2} \mu^c_{\langle u \rangle}(M)$$

$B_o(r) = \{x \in M : d(o, x) < r\}$ ($o \in M$: a reference point)

$$\lambda_{\mathrm{e}} \leq \begin{cases} \dfrac{1}{4} \left( \limsup\limits_{r \to \infty} \dfrac{1}{r} \log v(B_o(r)) \right)^2 & (v(M) = \infty) \\ \dfrac{1}{4} \left( \limsup\limits_{r \to \infty} \dfrac{-1}{r} \log v(M \setminus B_o(r)) \right)^2 & (v(M) < \infty) \end{cases}$$

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**Improvement by Kumura (2007):**

$$\lambda_{\mathrm{e}} \leq$$

$$\begin{cases} \dfrac{1}{4}\left(\liminf_{r\to\infty} \dfrac{1}{r}\log v(B_o(r))\right)^2 & (v(M) = \infty) \\ \dfrac{1}{4}\left(\liminf_{r\to\infty} \dfrac{-1}{r}\log v(M \setminus B_o(r))\right)^2 & (v(M) < \infty) \end{cases}$$

## Generalizations of Brook's results to regular Dirichlet forms

**Notarantonio (1998):**

**Strongly local regular Dirichlet form (intrinsic metric)**

**Folz (2014):**

**Dirichlet forms on weighted graphs (adapted length)**

**Haeseler-Keller-Wojciechowski (2013):**

**Regular Dirichlet forms with no killing part**

**(intrinsic metric in the sense of Frank-Lenz-Wingert (2014))**

## Applications: Strong locality or graph structure only

**Intrinsic metric conditions may force**

$$\sup_{x\in E}\int_{E\setminus\{x\}} d(x,y)^2 J(x,\mathrm{d}y) < \infty \quad \cdots \ (\star)$$

**provided that** $J(\mathrm{d}x,\mathrm{d}y) = J(x,\mathrm{d}y)m(\mathrm{d}x)$**.**

$(\star)$ **fails for the fractional Laplacian** $-(-\Delta)^{\alpha/2}$ $(\alpha \in (0,2))$**.**

**A natural substitute for** $(\star)$ **is**

$$\begin{cases} \sup_{x\in E}\int_{d(x,y)<1} d(x,y)^2 J(x,\mathrm{d}y) < \infty, \\ \sup_{x\in E}\int_{d(x,y)\geq 1} J(x,\mathrm{d}y) < \infty. \end{cases}$$

**S.-J. Wang (2023), S. (2023):**

**Necessary and sufficient condition for compactness/transience in terms of the coefficient growth for small and big jumps**

**Purpose in this talk:**

**To understand how the big jump affects the spectral structure.**

**Approach:**

- **to apply the previous argument to the small jump part**
- **to dominate the big jump part by the $L^2$-norm**

$(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2 = L^2(E; m)$

$C_0(E)$: totality of conti. functions with compact support in $E$

**Assumption 2.1 (S. (2015/2016))**

**(i)** $\forall u \in \mathcal{F} \cap C_0(E)$,

$$\mathcal{E}(u,u) = \iint_{E \times E \setminus \mathrm{diag}} (u(x) - u(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$

$J(x, \mathrm{d}y)$: positive measurable kernel on $(E, \mathcal{B}(E))$ s.t.

$$J(x, \mathrm{d}y) m(\mathrm{d}x) = J(y, \mathrm{d}x) m(\mathrm{d}y)$$

**(ii)** $\exists\{\rho_r\}_{r>0} \subset \mathcal{F}_{\rm loc} \cap C(E)$, $\exists\{F_r\}_{r>0} \subset C(E\times E)$ **with**

- $\forall (x,y) \in E\times E$, $\forall r>0$, $F_r(x,y)=F_r(y,x)>0$
- $\forall (x,y) \in E\times E$, $F_r(x,y) \nearrow$ **as** $r \nearrow$

**s.t. the following are finite for each** $r>0$**.**

$$M_1(r) := \operatorname*{ess\,sup}_{x\in E} \int_{d(x,y)<F_r(x,y)} (\rho_r(x)-\rho_r(y))^2 J(x,\mathrm{d}y)$$

$$M_2(r) := \operatorname*{ess\,sup}_{x\in E} \int_{d(x,y)\geq F_r(x,y)} J(x,\mathrm{d}y)$$

**(iii)** $\forall r>0$, $\forall R>0$, $K_{\rho_r}(R) := \{x\in E \mid \rho_r(x)\leq R\}$ **is compact, and for each** $r>0$, $K_{\rho_r}(R) \nearrow E$ **as** $R\nearrow\infty$.

## Theorem 2.2

**(1) If $m(E) = \infty$, then**

$$\lambda_e \le \inf_{r>0} \left( \frac{\mu_r^2}{4} M_1(r) + 2M_2(r) \right)$$

**with $\mu_r = \liminf_{R\to\infty} \frac{1}{R} \log m(K_{\rho_r}(R))$.**

**(2) If $m(E) < \infty$ and $(\mathcal{E}, \mathcal{F})$ is recurrent ($\rightsquigarrow 1 \in \mathcal{F}$), then**

$$\lambda_e \le \inf_{r>0} \left( \frac{\nu_r^2}{4} M_1(r) + 2M_2(r) \right)$$

**with $\nu_r = \liminf_{R\to\infty} \frac{-1}{R} \log m(E \setminus K_{\rho_r}(R))$.**

## 3. Sketch of the proof of Theorem 2.2 (1)

**Proposition 3.1 (Persson's formula)**

**If** $\exists\{f_n\} \subset \mathcal{F}$ **with** $\|f_n\|_{L^2} = 1$ $(\forall n \geq 1)$ **s.t.** $f_n \to 0$ $L^2$**-weakly, then**

$$\lambda_{\mathrm{e}} \leq \liminf_{n\to\infty} \mathcal{E}(f_n, f_n).$$

**Haeseler-Keller-Wojciechowski (2013)**

**Keller-Lenz-Wojciechowski (2021)**

In what follows, we suppose that $m(E) = \infty$.

Recall $\mu_r = \liminf_{R \to \infty} \frac{1}{R} \log m(K_{\rho_r}(R))$ $(r > 0)$

$\Rightarrow \forall r > 0$, $\exists \{R_n\}$ s.t.

$$\mu_r = \lim_{n \to \infty} \frac{1}{R_n} \log m(K_{\rho_r}(R_n))$$

For $r > 0$ and $\alpha > \mu_r/2$, we define

$$f_n^{(r)}(x) = \begin{cases} e^{\alpha R_n/2} - 1 & \left(0 \le \rho_r(x) \le \dfrac{R_n}{2}\right), \\ e^{\alpha(R_n - \rho_r(x))} - 1 & \left(\dfrac{R_n}{2} < \rho_r(x) \le R_n\right), \\ 0 & (\rho_r(x) > R_n). \end{cases}$$

## Lemma 3.2

For each $r > 0$, $\{f_n^{(r)}\} \subset \mathcal{F} \cap C_0(E)$ and $\dfrac{f_n^{(r)}}{\|f_n^{(r)}\|_{L^2}} \to 0$ $L^2$-weakly. In particular,

$$\lambda_{\mathrm{e}} \leq \liminf_{n\to\infty} \frac{1}{\|f_n^{(r)}\|_{L^2}^2}\mathcal{E}(f_n^{(r)}, f_n^{(r)}).$$

$$\begin{aligned}
&\mathcal{E}(f_n^{(r)}, f_n^{(r)}) \\
&= \iint_{E\times E\setminus\mathrm{diag}} (f_n^{(r)}(x) - f_n^{(r)}(y))^2\, J(x, \mathrm{d}y)m(\mathrm{d}x) \\
&= \iint_{d(x,y)<F_r(x,y)} (f_n^{(r)}(x) - f_n^{(r)}(y))^2\, J(x, \mathrm{d}y)m(\mathrm{d}x) \\
&+ \iint_{d(x,y)\geq F_r(x,y)} (f_n^{(r)}(x) - f_n^{(r)}(y))^2\, J(x, \mathrm{d}y)m(\mathrm{d}x)
\end{aligned}$$

$$g_n^{(r)}(x) := (f_n^{(r)}(x) + 2)1_{K_{\rho_r(R_n)}}(x)$$

**Using**

$$(f_n^{(r)}(x) - f_n^{(r)}(y))^2 \le \frac{\alpha^2}{2}(\rho_r(x) - \rho_r(y))^2 (g_n^{(r)}(x)^2 + g_n^{(r)}(y)^2),$$

**and the symmetry of** $J(x, \mathrm{d}y)m(\mathrm{d}x)$**, we have**

$$\begin{aligned}
&\iint_{d(x,y)<F_r(x,y)} (f_n^{(r)}(x) - f_n^{(r)}(y))^2 \, J(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\le \alpha^2 \iint_{d(x,y)<F_r(x,y)} (\rho_r(x) - \rho_r(y))^2 \\
&\qquad\qquad \times g_n^{(r)}(x)^2 \, J(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\le \alpha^2 M_1(r) \int g_n^{(r)}(x)^2 \, m(\mathrm{d}x) = \alpha^2 M_1(r) \|g_n^{(r)}\|_{L^2}^2.
\end{aligned}$$

Since

$$\iint_{d(x,y)\geq F_r(x,y)} (f_n^{(r)}(x) - f_n^{(r)}(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$
$$\leq 2M_2(r)\|f_n^{(r)}\|_{L^2}^2,$$

we obtain

$$\mathcal{E}(f_n^{(r)}, f_n^{(r)}) \leq \alpha^2 M_1(r)\|g_n^{(r)}\|_{L^2}^2 + 2M_2(r)\|f_n^{(r)}\|_{L^2}^2.$$

As we can show $\|f_n^{(r)}\|_{L^2}/\|g_n^{(r)}\|_{L^2} \to 1$ $(n \to \infty)$, we have

$$\frac{1}{\|f_n^{(r)}\|_{L^2}^2}\mathcal{E}(f_n^{(r)}, f_n^{(r)}) \leq \alpha^2 M_1(r)\frac{\|g_n^{(r)}\|_{L^2}^2}{\|f_n^{(r)}\|_{L^2}^2} + 2M_2(r)$$
$$\to \alpha^2 M_1(r) + 2M_2(r).$$

Hence by Proposition 3.1 (Persson's formula),

$$\lambda_{\mathrm{e}} \le \limsup_{n\to\infty} \frac{1}{\|f_n^{(r)}\|_{L^2}^2}\mathcal{E}(f_n^{(r)}, f_n^{(r)}) \le \alpha^2 M_1(r) + 2M_2(r).$$

As $r > 0$ and $\alpha > \mu_r/2$ are arbitrary, we get

$$\lambda_{\mathrm{e}} \le \inf_{r>0} \left( \frac{\mu_r^2}{4} M_1(r) + 2M_2(r) \right).$$

The proof is complete.

## 4. Applications (1)

**In what follows, we assume that**

- $J(x, \mathrm{d}y) = J(x, y)\, m(\mathrm{d}y)$;
- $\exists o \in E$, $d_o(x) := d(x, o) \in \mathcal{F}_{\mathrm{loc}}$;
- **Any closed ball in $E$ is compact.**

$$\mathcal{E}(u, u) = \iint_{E \times E \setminus \mathrm{diag}} (u(x) - u(y))^2 J(x, y) m(\mathrm{d}x) m(\mathrm{d}y)$$

## Example 4.1 (Exponential volume growth)

**Suppose that** $m(E)=\infty$, $\exists\eta>0$, $\exists\kappa>0$, $\exists\alpha\in(0,2)$ **s.t.**

$$m(B_x(R))\lesssim\begin{cases}R^{\eta} & (0<R<1),\\ e^{\kappa R} & (R\geq 1)\end{cases}\qquad (x\in E)$$

**and**

$$J(x,y)\lesssim\begin{cases}\dfrac{1}{d(x,y)^{\eta+\alpha}} & (0<d(x,y)<1),\\ \dfrac{e^{-\kappa d(x,y)}}{d(x,y)^{1+\alpha/2}} & (d(x,y)\geq 1).\end{cases}$$

**For $r > 0$, we take $\rho_r(x) = d_o(x)$ and $F_r(x, y) = r$. Then**

$$M_1(r) = \operatorname*{ess\,sup}_{x \in E} \int_{d(x,y)<r} (d_o(x) - d_o(y))^2 J(x, y)\, m(\mathrm{d}y)$$

$$\lesssim \begin{cases} r^{2-\alpha} & (0 < r < 1), \\ r^{2-\alpha/2} & (r \geq 1), \end{cases}$$

$$M_2(r) = \operatorname*{ess\,sup}_{x \in E} \int_{d(x,y)\geq r} J(x, y)\, m(\mathrm{d}y)$$

$$\lesssim \begin{cases} r^{-\alpha} & (0 < r < 1), \\ r^{-\alpha/2} & (r \geq 1). \end{cases}$$

$$M_1(r) \lesssim \begin{cases} r^{2-\alpha} & (0 < r < 1), \\ r^{2-\alpha/2} & (r \geq 1), \end{cases} \qquad M_2(r) \lesssim \begin{cases} r^{-\alpha} & (0 < r < 1), \\ r^{-\alpha/2} & (r \geq 1). \end{cases}$$

**We thus have for some $c > 0$,**

$$\lambda_{\mathrm{e}} \leq \inf_{r>0} \left( \frac{\kappa^2}{4} M_1(r) + 2M_2(r) \right) \leq c\kappa^{\alpha}.$$

**Remark. (Comparison with the $n$-dim. hyperbolic space $\mathbb{H}^n$)**

**(i) $\Delta$: Laplace-Beltrami operator on $\mathbb{H}^n$**

**For $-(-\Delta)^{\alpha/2}$ ($\alpha \in (0,2)$), all the conditions in Example 4.1 are fulfilled with $\eta = n$ and $\kappa = n-1$.**

**Grigor'yan-Huang-Masamune (2012), Ryznar-Żak (2016)**

$\circ \inf \sigma_{\mathrm{ess}}\left(-(-\Delta)^{\alpha/2}\right) = \dfrac{(n-1)^\alpha}{2^\alpha}$

$\circ\, m(B(R)) \sim c_n e^{(n-1)R}$

**However, the constant $c$ in our bound $\lambda_{\mathrm{e}} \leq c(n-1)^\alpha$ may depend on the dimension $n$.**

**(ii) Haeseler et. al. (2013), Frank-Lenz-Wingert (2014)**

**$\forall p \in (0, \alpha/4)$, $\exists c_p > 0$, $\rho(x,y) := c_p(d(x,y) \wedge d(x,y)^p)$ is an intrinsic metric in the sense of FLW (2014). However,**

$$m(B_\rho(R)) \asymp e^{R^{1/p}} \rightsquigarrow \lim_{R\to\infty} \frac{1}{R} \log m(B_\rho(R)) = \infty$$

## Example 4.2 (Coefficient growth)

**Suppose that $m(E)=\infty$, $\exists\eta>0$, $\exists\beta\in(0,2)$ s.t.**

$$m(B_x(R))\lesssim R^{\eta}\quad (x\in E),\quad J(x,y)\le\frac{c(x,y)}{d(x,y)^{\eta+\beta}},$$

**where $p\ge 0$, $q\in[0,\beta)$ and**

$$c(x,y)=\{(1+d_o(x))^p+(1+d_0(y))^p\}1_{\{d(x,y)<1\}}$$
$$+\{(1+d_o(x))^q+(1+d_0(y))^q\}1_{\{d(x,y)\ge 1\}}$$

**a) $p<2\Rightarrow\lambda_{\rm e}=0$ b) $p=2\Rightarrow\lambda_{\rm e}<\infty$**

**($\rho_r(x)=\log(r+d_o(x))$, $F_r(x,y)=c\{r+(d_o(x)\vee d_o(y))\}$)**

**Remark. (Sharpness in regard to the positivity of $\lambda_{\mathrm{e}}$)**

$E = \mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$, $J(x, y) \asymp \dfrac{c(x, y)}{|x - y|^{d+\beta}}$

$p \geq 0$, $q \in [0, \beta)$ and

$$c(x, y) = \{(1 + |x|)^p + (1 + |y|)^p\} \mathbf{1}_{\{|x-y|<1\}} + \{(1 + |x|)^q + (1 + |y|)^q\} \mathbf{1}_{\{|x-y|\geq 1\}}$$

$\lambda_{\mathrm{e}} < \infty \iff p \leq 2$ (S.-J. Wang (2023))

a') $p < 2 \Rightarrow \lambda_{\mathrm{e}} = 0$

b') $p = 2 \Rightarrow \lambda_{\mathrm{e}} \in (0, \infty)$ (Lenz-Stollmann (2019), S. (2023))

## 5. Applications (2)

**Example 5.1** **(OU type, F.-Y. Wang and J. Wang (2015))**

**Suppose that** $\exists \eta > 0$ **and** $\beta \in (0,2)$ **s.t.**

$$m(B_x(r)) \asymp r^\eta \ (x \in E), \quad J(x,y) \lesssim \frac{1}{d(x,y)^{\eta+\beta}}$$

$V(r)$: **positive increasing function on** $[0,\infty)$

$\mu_V(\mathrm{d}x) := e^{-V(d_o(x))} m(\mathrm{d}x) \qquad (d_o(x) = d(o,x))$

$$\mathcal{E}(u,u) = \iint_{E\times E\setminus \mathrm{diag}} (u(x)-u(y))^2 J(x,y)\, m(\mathrm{d}y)\mu_V(\mathrm{d}x)$$

$$\mathcal{D}(\mathcal{E}) = \{u \in L^2(E;\mu_V) : \mathcal{E}(u,u) < \infty\}$$

$$\mathcal{E}(u,u) = \iint_{E\times E\setminus \mathrm{diag}} (u(x)-u(y))^2 J(x,y)\, m(\mathrm{d}y)\mu_V(\mathrm{d}x)$$

$$\mathcal{D}(\mathcal{E}) = \{u \in L^2(E;\mu_V) : \mathcal{E}(u,u) < \infty\}$$

**Suppose also that $\mu_V(E) < \infty$**

**$\Rightarrow$ $(\mathcal{E},\mathcal{F})$ is recurrent ($\mathcal{F}$: the closure of $C_b^{\mathrm{lip}}(E)$ in $\mathcal{D}(\mathcal{E})$)**

$$\nu_r := \liminf_{R\to\infty} \frac{-1}{R} \log m(E \setminus K_{\rho_r}(R))$$

**If $\exists\ \delta > 0$, $e^{V(r)}/e^{V(s)} \lesssim (r/s)^\delta$ $(0 < s < r)$, then**

$$\lambda_{\mathrm{e}} \lesssim \limsup_{r\to\infty} \left(\nu_r r^{2-\beta} e^{V(r)} + r^{-(\eta+\beta)} e^{V(r)}\right)$$

**$(\rho_r(x) = d_o(x)$, $F_r(x,y) = r)$**

**For instance, if $E = \mathbb{R}^d$ and $m$ is the Lebesgue measure, then**

$$\lambda_{\mathrm{e}} \lesssim \limsup_{r\to\infty} \left( r^{-(\eta+\beta)} e^{V(r)} \right)$$

**If we assume in addition that**

$$J(x,y) \asymp \frac{1}{|x-y|^{\eta+\beta}},$$

**then by F.-Y. Wang and J. Wang (2015),**

$$\limsup_{r\to\infty} \left( r^{-(\eta+\beta)} e^{V(r)} \right) = \infty \Rightarrow \lambda_{\mathrm{e}} = \infty$$