Maximal displacement of branching symmetric stable processes

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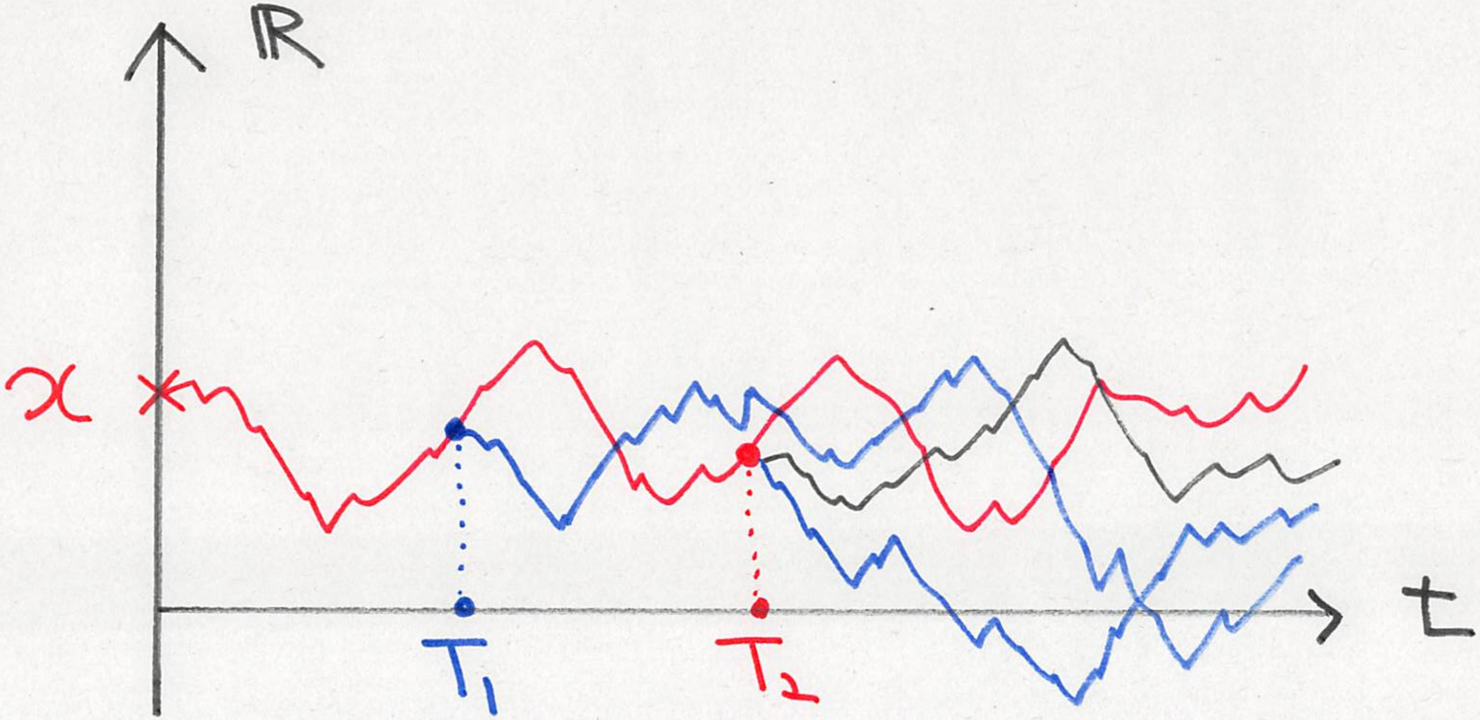
1. Introduction



<u>Subject in this talk:</u> **Bounds of the range of particles**

- particle motions
- branching mechanism (splitting time and offspring number)

- $\{X_n\}_{n=1}^{\infty}$: nonnegative i.i.d. random variables
- $L_n := \max_{1 \le k \le n} X_k$

Assume that for some $\alpha > 0$,

$$P(X_1 > \kappa) = \frac{1}{\kappa^{\alpha}} \quad (\kappa \ge 1)$$

Noting that $P(L_n \le \kappa) = P(X_1 \le \kappa)^n$, we have

$$\lim_{n \to \infty} P\left(\frac{L_n}{n^{1/\alpha}} \le \kappa\right) = \exp\left(-\frac{1}{\kappa^{\alpha}}\right)$$

[Durrett(19), Exercise 3.2.2]

- $F(\kappa) = \exp\left(-1/\kappa^{\alpha}\right)$: Fréchet distribution

○ **Branching symmetric stable process on $\mathbb{R}^d$**

▷ $\{X_t\}_{t\geq 0}$: **symm. stable process on** $\mathbb{R}^d$

generated by $-(-\Delta)^{\alpha/2}/2$ **(**$0 < \alpha < 2$**)**

Then for some $c_0 > 0$, $P_x(|X_t| > \kappa) \sim c_0 t/\kappa^\alpha$ $(\kappa \to \infty)$

▷ V: **nonnegative function on** $\mathbb{R}^d$

The splitting time T **is distributed as**

$$\mathbb{P}_x(t < T \mid \sigma(X)) = \exp\left(-\int_0^t V(X_s)\,\mathrm{d}s\right)$$

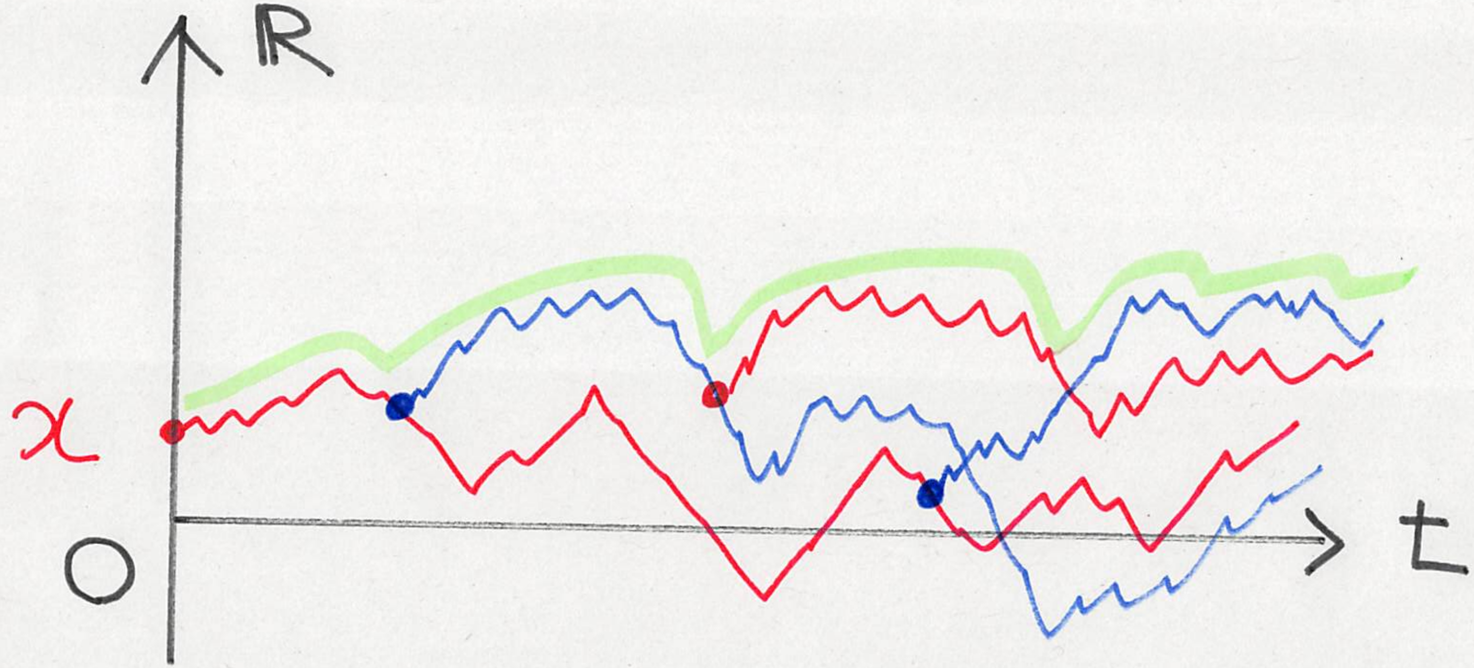
▷ $\mathrm{p} = \{p_n(x)\ (x \in \mathbb{R}^d)\}_{n=0}^{\infty}$: **offspring distribution**

▷ Z_t: the number of particles alive at time t

▷ $\mathbf{X}_t = (\mathbf{X}_t^1, \ldots, \mathbf{X}_t^{Z_t}) (\in (\mathbb{R}^d)^{Z_t})$: particle positions at time t

▷ $L_t := \max_{1 \le k \le Z_t} |\mathbf{X}_t^k|$:

maximal Euclidean norm of particles alive at time t



▷ $Q(x) := \sum_{n=0}^{\infty} n p_n(x)$: expected offspring number

▷ $W(x) := (Q(x) - 1)V(x)$: intensity of branching

Assumption.

(1) V is a Kato class function with compact support in $\mathbb{R}^d$

(2) p has the uniform finite second moment

(3) $\lambda := \inf \mathrm{Spec}\left(\frac{1}{2}(-\Delta)^{\alpha/2} - W\right) < 0$

Theorem 1. $\exists c_* > 0$, $\exists$ nonnegative r.v. M_∞, $\forall \kappa > 0$,

$$\lim_{t\to\infty} \mathbb{P}_x\left(\frac{L_t}{(e^{-\lambda t})^{1/\alpha}} \le \kappa\right) = \mathbb{E}_x\left[\exp\left(-\frac{c_* M_\infty}{\kappa^\alpha}\right)\right]$$

▷ Z: nonnegative Fréchet distributed r.v.

Then by Theorem 1,

$$\lim_{t\to\infty} \mathbb{P}_x\left(\frac{L_t}{(e^{-\lambda t})^{1/\alpha}} \le \kappa\right) = \mathbb{E}_x\left[\exp\left(-\frac{c_* M_\infty}{\kappa^\alpha}\right)\right]$$

$$= \int_0^\infty \mathbb{P}_x(M_\infty \in \mathrm{d}s) P\left((c_* s)^{1/\alpha} Z \le \kappa\right)$$

$$\left[\lim_{n\to\infty} P\left(\frac{L_n}{n^{1/\alpha}} \le \kappa\right) = P(Z \le \kappa) \text{ for } L_n = \max_{1\le k\le n} X_k\right]$$

- Branching RW on $\mathbb{Z}$ with spatially homogeneous branching

  [Durrett(83), Bhattacharya-Hazra-Roy(17)]

- (conti. time) Catalytic branching RW on $\mathbb{Z}$ [Bulinskaya(21)]

2. Results $(V(x)\,\mathrm{d}x \rightsquigarrow \mu(\mathrm{d}x))$

▷ μ: positive Radon measure on $\mathbb{R}^d$ with compact support

▷ $G_\beta(x,y)$: β-resolvent of the symm. α-stable proc. on $\mathbb{R}^d$

Assume that μ belongs to the Kato class,

$$\lim_{\beta\to\infty} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x,y)\,\mu(\mathrm{d}y) = 0$$

▷ $\{A_t^\mu\}_{t\geq 0}$: positive continuous additive functional of μ

The splitting time T is distributed as

$$\mathbb{P}_x\,(t < T \mid \sigma(X)) = e^{-A_t^\mu}$$

▷ $L_t := \max_{1 \le k \le Z_t} |\mathrm{X}_t^k|$

▷ $\nu(\mathrm{d}x) = (Q(x) - 1)\,\mu(\mathrm{d}x)$: **intensity of branching**

Assumption.

(1) p **has the uniform finite second moment**

(2) $\lambda := \inf \mathrm{Spec}\left(\frac{1}{2}(-\Delta)^{\alpha/2} - \nu\right) < 0$

Lemma.

If $\lambda < 0$, then the ground state $h \in C_b^+(\mathbb{R}^d)$ exists and

$$h(x) \sim \frac{C_0}{|x|^{d+\alpha}} \int_{\mathbb{R}^d} h(y)\,\nu(\mathrm{d}y) \quad (|x| \to \infty)$$

▷ $M_t := e^{\lambda t} \sum_{k=1}^{Z_t} h(\mathrm{X}_t^k)$: **nonneg. square integrable martingale**

▷ $M_\infty := \lim_{t\to\infty} M_t \in [0,\infty)$

Theorem 1. $\exists c_* > 0$ **(explicit),** $\forall \kappa > 0$,

$$\lim_{t\to\infty} \mathbb{P}_x\left(\frac{L_t}{(e^{-\lambda t})^{1/\alpha}} \le \kappa\right) = \mathbb{E}_x\left[\exp\left(-\frac{c_* M_\infty}{\kappa^\alpha}\right)\right]$$

▷ $a(t)$: **positive m'ble funct. s.t.** $a(t) \to \infty$ **(**$t \to \infty$**)**

Theorem 2. $\exists c_* > 0$ **(as in Thm), loc. uniformly in** $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\left(\frac{L_t}{(e^{-\lambda t})^{1/\alpha}} > a(t)\right) \sim \frac{c_*}{a(t)^\alpha} h(x) \ (t \to \infty)$$

Remark. **(Degeneracy of M_∞ [S.(08, 19)])**

▷ $e_0 := \inf\{t > 0 \mid Z_t = 0\}$: **extinction time**

$\Rightarrow \{e_0 = \infty\} \supset \{M_\infty > 0\}$

▷ $\mathbb{P}^*_x(\cdot) := \mathbb{P}_x(\cdot \mid M_\infty > 0)$

- $d = 1$, $\alpha \in (1,2) \Rightarrow \{e_0 = \infty\} = \{M_\infty > 0\}$ **and**

$$\lim_{t\to\infty} \mathbb{P}^*_x\left(\frac{L_t}{\left(e^{-\lambda t}\right)^{1/\alpha}} \le \kappa\right) = \mathbb{E}^*_x\left[\exp\left(-\frac{c_* M_\infty}{\kappa^\alpha}\right)\right] \cdots (\star)$$

- $d > \alpha \Rightarrow \mathbb{P}_x(\{e_0 = \infty\} \cap \{M_\infty = 0\}) > 0$

In particular, the validity of $(\star)$ remains unknown.

3. Sketch of the proof of Theorem 1

We follow [Nishimori-S.(22), Bulinskaya(20)] to show

$$\lim_{t\to\infty} \mathbb{P}_x\left(L_t \leq \kappa e^{-\lambda t/\alpha}\right) = \mathbb{E}_x\left[\exp\left(-\frac{c_* M_\infty}{\kappa^\alpha}\right)\right]$$

▷ $R^\kappa(t) := \kappa e^{-\lambda t/\alpha}$ **(**$\kappa > 0$**: fixed)**

Since $\mathbb{P}_x(L_t < \infty) = 1$, $\forall \varepsilon > 0$, $\forall T_1 > 0$, $\exists r_1 > 0$, $\forall t \geq T_1$,

$$\mathbb{P}_x(L_t \leq R^\kappa(t)) \leq \mathbb{P}_x(L_t \leq R^\kappa(t), L_{T_1} \leq r_1) + \varepsilon$$

$$= \mathbb{E}_x\left[\mathbb{P}_{\mathrm{X}_{T_1}}(L_{t-T_1} \leq R^\kappa(t)); L_{T_1} \leq r_1\right] + \varepsilon$$

Note that on the event $\{L_{T_1} \leq r_1\}$, $|\mathrm{X}^k_{T_1}| \leq r_1$ **for any** k**.**

Since $R^{\kappa}(t) = \kappa e^{-\lambda t/\alpha}$ by definition and

$$\kappa e^{-\lambda t/\alpha} = \kappa e^{-\lambda T_1/\alpha} e^{-\lambda(t-T_1)/\alpha} = R^{\kappa e^{-\lambda T_1/\alpha}}(t - T_1),$$

we have by the branching property,

$$\begin{aligned}
&\mathbb{P}_{\mathbf{X}_{T_1}}(L_{t-T_1} \leq R^{\kappa}(t)) \\
&= \mathbb{P}_{\mathbf{X}_{T_1}}\left(L_{t-T_1} \leq R^{\kappa e^{-T_1\lambda/\alpha}}(t - T_1)\right) \\
&= \prod_{k=1}^{Z_{T_1}} \mathbb{P}_{\mathbf{X}^k_{T_1}}\left(L_{t-T_1} \leq R^{\kappa e^{-T_1\lambda/\alpha}}(t - T_1)\right) \\
&= \prod_{k=1}^{Z_{T_1}} \left[1 - \mathbb{P}_{\mathbf{X}^k_{T_1}}\left(L_{t-T_1} > R^{\kappa e^{-T_1\lambda/\alpha}}(t - T_1)\right)\right]
\end{aligned}$$

▷ $\mathcal{L} := \{L \subset \mathbb{R}^d : \text{compact}\}$

Lemma. **The following equality holds:**

$$\lim_{\kappa\to\infty} \sup_{L\in\mathcal{L}} \limsup_{t\to\infty} \sup_{x\in L} \left| \frac{\kappa^\alpha}{h(x)} \mathbb{P}_x(L_t > R^\kappa(t)) - c_* \right| = 0$$

Since $\forall\theta \in (0,1)$, $e^{-(1+\theta)x} \leq 1 - x \leq e^{-x}$ $(0 \leq x \ll 1)$,

$$\begin{aligned}
&\mathbb{P}_{\mathrm{X}_{T_1}}(L_{t-T_1} \leq R^\kappa(t)) \\
&= \prod_{k=1}^{Z_{T_1}} \left(1 - \mathbb{P}_{\mathrm{X}^k_{T_1}}\left(L_{t-T_1} > R^{\kappa e^{-T_1\lambda/\alpha}}(t - T_1)\right)\right) \\
&\simeq \prod_{k=1}^{Z_{T_1}} \exp\left(-\frac{c_*}{(\kappa e^{-T_1\lambda/\alpha})^\alpha} h(\mathrm{X}^k_{T_1})\right) = \exp\left(-\frac{c_*}{\kappa^\alpha} M_{T_1}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}_x(L_t \le R^\kappa(t)) &\simeq \mathbb{P}_x(L_t \le R^\kappa(t), L_{T_1} \le r_1) \\ &= \mathbb{E}_x\left[\mathbb{P}_{X_{T_1}}(L_{t-T_1} \le R^\kappa(t)); L_{T_1} \le r_1\right] \\ &\simeq \mathbb{E}_x\left[\exp\left(-\frac{c_*}{\kappa^\alpha} M_{T_1}\right); L_{T_1} \le r_1\right] \\ &\simeq \mathbb{E}_x\left[\exp\left(-\frac{c_*}{\kappa^\alpha} M_\infty\right)\right]\end{aligned}$$

■ Comment on the proof of Lemma

Lemma. The following equality holds:

$$\lim_{\kappa\to\infty} \sup_{L\in\mathcal{L}} \lim_{t\to\infty} \sup_{x\in L} \left| \frac{\kappa^\alpha}{h(x)} \mathbb{P}_x(L_t > R^\kappa(t)) - c_* \right| = 0$$

$\triangleright$ $Z_t^\kappa := \sharp$ of particles on $\{|x| > R^\kappa(t)\}$ at time t

Since $\{L_t > R^\kappa(t)\} = \{Z_t^\kappa(t) \geq 1\}$, we obtain

$$\begin{aligned}\mathbb{P}_x\left(L_t > R^\kappa(t)\right) &= \mathbb{P}_x(Z_t^\kappa \geq 1) \\ &\leq \mathbb{E}_x[Z_t^\kappa] = E_x\left[e^{A_t^\nu}; |X_t| > R^\kappa(t)\right]\end{aligned}$$

Recall $\nu = (Q(x) - 1)\mu$ (intensity of branching)

Proposition. $\forall \kappa > 0$, $\forall K \subset \mathbb{R}^d$ (compact),

$$\lim_{t\to\infty} \sup_{x\in K} \left| \frac{\kappa^\alpha}{h(x)} E_x\left[e^{A_t^\nu}; |X_t| > R^\kappa(t)\right] - c_* \right| = 0$$

Existence of the spectral gap [Z.-Q. Chen-S.(07)]

$\forall \kappa > 0,$

$$E_x\left[e^{A_t^\nu}; |X_t| > R^\kappa(t)\right] \sim e^{-\lambda t} h(x) \int_{|y|>R^\kappa(t)} h(y)\,\mathrm{d}y$$

$$\sim \frac{c_*}{\kappa^\alpha}$$

By the inequality

$$\mathbb{P}_x\left(L_t > R^\kappa(t)\right) = \mathbb{P}_x(Z_t^\kappa \geq 1) \geq \frac{\mathbb{E}_x[Z_t^\kappa]^2}{\mathbb{E}_x\left[(Z_t^\kappa)^2\right]}$$

and calculating $\mathbb{E}_x\left[(Z_t^\kappa)^2\right]$**, we have Lemma.**

Note: The value of c_* **is given by**

$$c_* = \frac{\sin(\pi\alpha/2)\Gamma((d+\alpha)/2)\Gamma(\alpha/2)}{(-\lambda)^2 2^{1-\alpha}\Gamma(d/2)} \int_{\mathbb{R}^d} h(y)\,\nu(\mathrm{d}y)$$

4. Examples

Assumption (2).

$$\lambda := \inf \mathrm{Spec}\left(\frac{1}{2}(-\Delta)^{\alpha/2} - \nu\right) < 0$$

Theorem 1. $\forall \kappa > 0$,

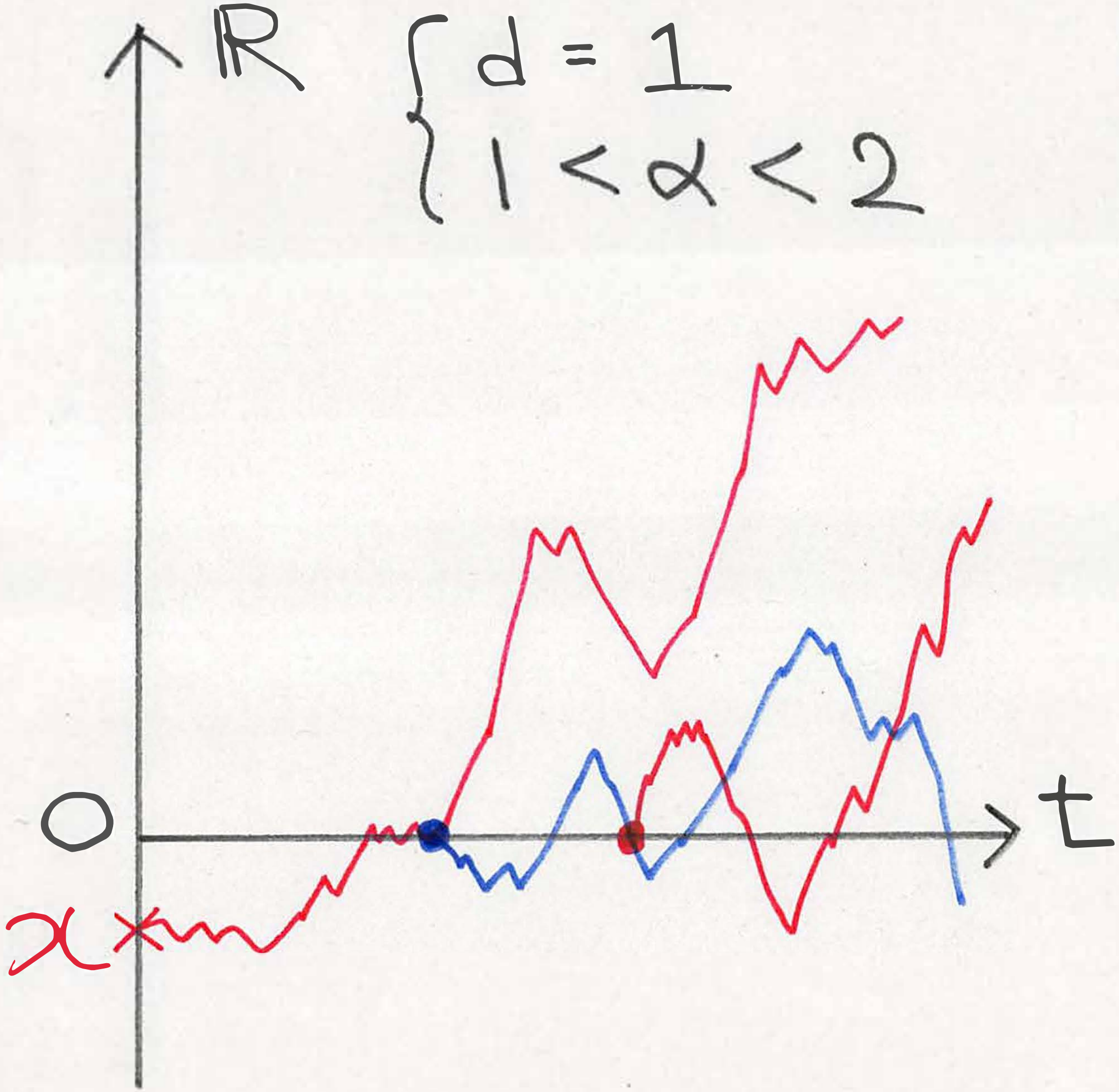
$$\lim_{t\to\infty} \mathbb{P}_x\left(\frac{L_t}{\left(e^{-\lambda t}\right)^{1/\alpha}} \leq \kappa\right) = \mathbb{E}_x\left[\exp\left(-\frac{c_* M_\infty}{\kappa^\alpha}\right)\right]$$

▷ $a(t)$ $(t > 0)$: positive m'ble funct., $a(t) \to \infty$ $(t \to \infty)$

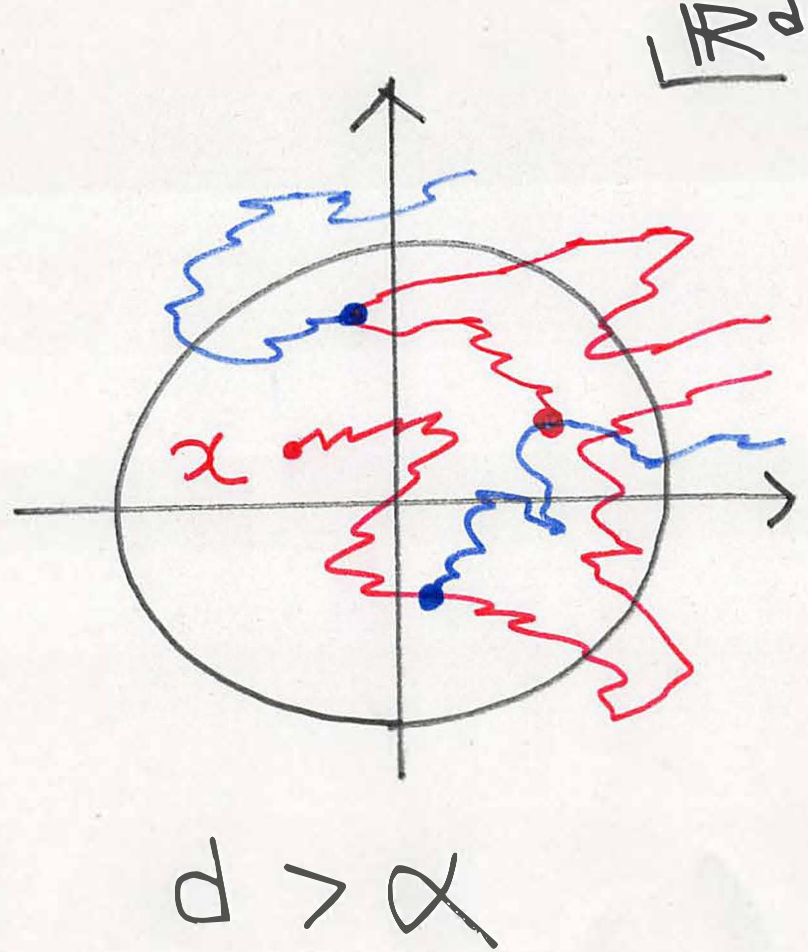
Theorem 2. Loc. uniformly in $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\left(\frac{L_t}{\left(e^{-\lambda t}\right)^{1/\alpha}} > a(t)\right) \sim \frac{c_*}{a(t)^\alpha} h(x) \quad (t \to \infty)$$

Example 1.



Example 2.



Example 1. $(d = 1,\ 1 < \alpha < 2)$

▷ $\mu = c\delta_0$ ($c > 0$, δ_0: Dirac meas. at the origin)

▷ $p_0 + p_2 = 1$, $m := 2p_2$ (expected offspring number)

$$(\mathbb{P}_x(e_0 = \infty) =)\ \mathbb{P}_x(M_\infty > 0) > 0 \iff p_2 > \frac{1}{2}$$

∘ $\lambda = \inf \mathrm{Spec}\left(\frac{1}{2}(-\Delta)^{\alpha/2} - c(m-1)\delta_0\right)$

By [S.(08)], we know that

$$\lambda = -\left\{\frac{c(m-1)2^{1/\alpha}}{\sin(\pi/\alpha)}\right\}^{\alpha/(\alpha-1)}, \quad h(x) = G_{-\lambda}(x,0)h(0)$$

Theorems 1 and 2 hold under this setting.

Example 2. $(d > \alpha)$

▷ $\mu(\mathrm{d}x) = c1_{\{|x|<r\}}\,\mathrm{d}x \ (c, r > 0)$, $p_2(x) \equiv 1$

∘ $\lambda = \inf \mathrm{Spec}\left(\frac{1}{2}(-\Delta)^{\alpha/2} - c1_{\{|x|<r\}}\right)$

$\Rightarrow \exists \kappa_1, \kappa_2 > 0 \ (0 < \kappa_1 < \kappa_2)$,

(a) $\quad r \leq \dfrac{\kappa_1}{c^{1/\alpha}} \Rightarrow \lambda \geq 0$

(b) $\quad r > \dfrac{\kappa_2}{c^{1/\alpha}} \Rightarrow \lambda < 0$

[S.-Takeda(05), Takeda-Tsuchida(07)]

Theorems 1 and 2 hold if $r > \kappa_2/c^{1/\alpha}$.