Symmetric non-local Dirichlet forms with singular coefficients: compactness and transience

塩沢 裕一 (大阪大学)

関西大学確率論研究会

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This talk is based on the following works:

1. **Y.S.-Jian Wang (Fujian Normal University), Compactness of semigroups generated by symmetric non-local Dirichlet forms with unbounded coefficients, to appear in Potential Anal., published online.**
2. **Y.S., Transience of symmetric non-local Dirichlet forms, to appear in Math. Nachr.**

1. Introduction and results

▷ $\alpha, \beta \in (0,2)$: fixed

▷ $C_0^\infty(\mathbb{R}^d)$: totality of smooth functions on $\mathbb{R}^d$ with cpt. supp.

For $u \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{E}(u,u) = &\iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} c_1(x,y)\,\mathrm{d}x\mathrm{d}y \\ &+ \iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}} c_2(x,y)\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$\begin{cases} c_1(x,y) \asymp (1+|x|^2)^{p/2} + (1+|y|^2)^{p/2} & (p \in \mathbb{R}) \\ c_2(x,y) \asymp (1+|x|^2)^{q/2} + (1+|y|^2)^{q/2} & (q < \beta) \end{cases}$$

$$\mathcal{E}(u,u) \asymp \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|^2)^{p/2}+(1+|y|^2)^{p/2}]\,\mathrm{d}x\mathrm{d}y$$
$$+\iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}}[(1+|x|^2)^{q/2}+(1+|y|^2)^{q/2}]\,\mathrm{d}x\mathrm{d}y$$

$\underset{[q<\beta]}{\rightsquigarrow}$ $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)(= L^2(\mathbb{R}^d; \mathrm{d}x))$

▷ $(\mathcal{E}, \mathcal{F})$: closure of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d)$

($\rightsquigarrow$ irreducible regular Dirichlet form on $L^2(\mathbb{R}^d)$)

∘ $p = q = 0$, $\alpha = \beta \Rightarrow (\mathcal{E}, \mathcal{F})$ is of symmetric α-stable type:

$$\mathcal{E}(u,u) \asymp \left(\frac{1}{2}(-\Delta)^{\alpha/2}u, u\right)_{L^2(\mathbb{R}^d)}$$

Compactness and transience

▷ $\{T_t\}_{t\geq 0}$: Markov semigroup on $L^2(\mathbb{R}^d)$ generated by $(\mathcal{E}, \mathcal{F})$

(a) T_t is compact on $L^2(\mathbb{R}^d)$

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall B \subset L^2(\mathbb{R}^d)$ (bdd.), $T_t B$ is rel. compact in $L^2(\mathbb{R}^d)$

▷ $(L, D(L))$: L^2-generator of $(\mathcal{E}, \mathcal{F})$ $(\mathcal{E}(u, v) = (-Lu, v))$

▷ $\sigma_{\text{ess}}(L)$: essential spectrum of L

⇝ $T_t (= e^{tL})$ is cpt for some/any $t > 0$

$\Longleftrightarrow$ $\sigma_{\text{ess}}(L) = \emptyset$, i.e., the spectrum of L is discrete

(b) $\{T_t\}_{t>0}$ **(or $\mathcal{E}$) is transient**

$$\underset{\text{def}}{\iff} \forall f \in L^2(\mathbb{R}^d)\ (f \geq 0 \text{ a.e.}), \int_0^\infty T_t f \, \mathrm{d}t < \infty, \text{a.e.}$$

$\triangleright$ $\mathrm{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}\right)$**: symm. Hunt process on** $\mathbb{R}^d$**:**

$$T_t f(x) = E_x[f(X_t)] \text{ q.e. } x \in \mathbb{R}^d,\ f \in L^2(\mathbb{R}^d) \cap \mathcal{B}_b(\mathbb{R}^d)$$

$\Rightarrow \forall A \in \mathcal{B}(\mathbb{R}^d)\ (|A| < \infty),$

$$\int_0^\infty T_t 1_A(x) \, \mathrm{d}t = E_x\left[\int_0^\infty 1_A(X_t) \, \mathrm{d}t\right]$$

$*$ **RHS: the expected occupation time of the set** A

$$\mathcal{E}(u,u) \asymp \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|^2)^{p/2}+(1+|y|^2)^{p/2}]\,\mathrm{d}x\mathrm{d}y$$
$$+ \iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}}[(1+|x|^2)^{q/2}+(1+|y|^2)^{q/2}]\,\mathrm{d}x\mathrm{d}y$$

Theorem. **Assume that $p \in \mathbb{R}$ and $q < \beta$.**

(1) [S.-J. Wang (21+)]

$\exists/\forall t > 0$, T_t **is compact on** $L^2(\mathbb{R}^d) \iff p > 2$

(2) [⇒: Ôkura-Uemura (15), Okamura-Uemura (21);

⇐: S. (21+)]

$\{T_t\}_{t>0}$ **is transient** $\iff p > 2-d$ **or** $(\beta >)\ q > \beta - d$

2. Proof of Theorem (1)

$$\mathcal{E}(u,u) \asymp \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|^2)^{p/2}+(1+|y|^2)^{p/2}]\,\mathrm{d}x\mathrm{d}y$$
$$+\iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}}[(1+|x|^2)^{q/2}+(1+|y|^2)^{q/2}]\,\mathrm{d}x\mathrm{d}y$$

Theorem. **(1) [S.-J. Wang (21+)]**

$\exists/\forall t>0$, T_t **is compact on** $L^2(\mathbb{R}^d) \iff p>2$

Proof of Theorem (1) $\Leftarrow$).

∘ **F.-Y. Wang (02, 05)**

T_t **is compact** $\underset{\text{[if } T_t \text{ has density]}}{\iff}$ **Super Poincaré inequality**

Super Poincaré inequality [F.-Y. Wang (02, 05)]

$$\begin{cases} \exists/\forall \psi \in L^2(\mathbb{R}^d) \ (\psi > 0), \\ \exists \beta : (0,\infty) \to (0,\infty)\text{: **decreasing**,} \end{cases}$$

$\forall r > 0, \forall u \in \mathcal{F}$,

$$\int_{\mathbb{R}^d} u^2 \, dx \le r\mathcal{E}(u,u) + \beta(r) \left(\int_{\mathbb{R}^d} |u|\psi \, dx \right) \cdots (\star)$$

$*$ $(\star)$ **provides a lower bound of** $\inf \sigma_{\mathrm{ess}}(L)$:

$\exists r_0 > 0, \forall r > r_0, \forall u \in \mathcal{F}$, $(\star)$ **holds** $\iff \inf \sigma_{\mathrm{ess}}(L) \ge r_0^{-1}$

$\circ \displaystyle\int_{|x| \ge R_0} u^2 \, dx \le \cdots$

$p > 2 \Rightarrow \forall \lambda > 0, \exists V_\lambda \in C^1_{++}(\mathbb{R}^d), \exists R_0 > 0,$

- $(1+|x|^2)^{p/2} + (1+|y|^2)^{p/2} \geq V_\lambda(x) + V_\lambda(y) \ (|x-y| < 1)$
- $V_\lambda(x) = \lambda(1+|x|^2) \ (|x| \geq R_0)$

Hence

$$\mathcal{E}(u,u) \geq \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[V_\lambda(x)+V_\lambda(y)]\,\mathrm{d}x\mathrm{d}y$$

$$=: \mathcal{E}^\lambda(u,u)$$

▷ $\phi(x) := (1+|x|^2)^{-\delta/2} \ (\delta \in (0,1))$

$$\mathcal{E}^\lambda(u,u) \geq \mathcal{E}^\lambda(\phi, u^2/\phi) = 2\int_{\mathbb{R}^d} \frac{-\mathcal{L}_\lambda \phi}{\phi} u^2\,\mathrm{d}x \quad \text{(Hardy-type)}$$

$$\mathcal{L}_\lambda \phi(x) = \int_{0<|z|<1} (\phi(x+z) - \phi(x) - \langle \nabla \phi(x), z \rangle) \frac{V_\lambda(x) + V_\lambda(x+z)}{|z|^{d+\alpha}} \, \mathrm{d}z + \frac{1}{2} \int_{0<|z|<1} \langle \nabla \phi(x), z \rangle \frac{V_\lambda(x+z) - V_\lambda(x-z)}{|z|^{d+\alpha}} \, \mathrm{d}z$$

Calculation similar to [S.-Uemura (14)]

Proposition. $\forall C > 0, \exists \lambda > 0, \exists R_0 > 0, \exists C_0 > 0,$

$$\boxed{\frac{-\mathcal{L}_\lambda \phi}{\phi}(x) \geq C 1_{\{|x| \geq R_0\}}(x) - C_0 1_{\{|x| < R_0\}}(x) \ (\forall x \in \mathbb{R}^d)}$$

$$\mathcal{E}(u,u) \geq 2 \int_{\mathbb{R}^d} \frac{-\mathcal{L}_\lambda \phi}{\phi} u^2 \, \mathrm{d}x \underset{\text{Prop.}}{\geq} \underline{2C \int_{|x| \geq R_0} u^2 \, \mathrm{d}x} - 2C_0 \int_{|x| < R_0} u^2 \, \mathrm{d}x$$

Namely,

$$\int_{|x|\geq R_0} u^2\,\mathrm{d}x \leq \frac{1}{2C}\mathcal{E}(u,u) + \frac{C_0}{C}\boxed{\int_{|x|<R_0} u^2\,\mathrm{d}x}$$

Upper bound of $\boxed{\cdots}$ [Chen-Kim-Kumagai (08)]:

Nash inequality for the truncated symmetric stable process

($\rightsquigarrow \beta(r)$ comes from the heat kernel upper bound)

We finally take $r = C^{-1}$ so that

$$\int_{\mathbb{R}^d} u^2\,\mathrm{d}x \leq r\mathcal{E}(u,u) + \beta(r)\left(\int_{\mathbb{R}^d} |u|\psi\,\mathrm{d}x\right)\cdots(\star)$$

for any $\psi \in L^2(\mathbb{R}^d) \cap \mathcal{B}_b(\mathbb{R}^d)$ $(\psi > 0)$

3. Proof of Theorem (2)

$$\mathcal{E}(u,u) \asymp \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|^2)^{p/2}+(1+|y|^2)^{p/2}]\,dxdy$$
$$+\iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}}[(1+|x|^2)^{q/2}+(1+|y|^2)^{q/2}]\,dxdy$$

Theorem (a part of) (2). [S. (21+)]

$p > 2-d$ or $(\beta >)$ $q > \beta - d \Rightarrow \{T_t\}_{t>0}$ is transient

Proof.

Lyapunov method and comparison principle

[Kolokoltsov-Schilling-Tyukov (02)]

Comparison principle

For $u \in C_0^\infty(\mathbb{R}^d)$, define

$$\mathcal{A}(u,u) = \iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\beta}}[(1+|x|^2)^{q/2} + (1+|y|^2)^{q/2}]\,\mathrm{d}x\mathrm{d}y$$

▷ $(\mathcal{A},\mathcal{G})$: closure of $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d)$

$\Rightarrow$ $(\mathcal{A},\mathcal{G})$ is an irreducible regular Dirichlet form on $L^2(\mathbb{R}^d)$,

$$\mathcal{F} \subset \mathcal{G} \text{ and } \exists c > 0, \mathcal{E}(u,u) \geq c\mathcal{A}(u,u) \ (u \in \mathcal{F})$$

$\rightsquigarrow$ "$\mathcal{A}$ is non-recurrent $\Rightarrow$ $\mathcal{A}$ is transient $\Rightarrow$ $\mathcal{E}$ is transient"

[Ôkura (92), Fukushima-Oshima-Takeda (11)]

Lyapunov method [Pinsky (95)]

For any $u \in C_0^\infty(\mathbb{R}^d)$, $\mathcal{A}(u,u) = (-\mathcal{L}u, u)_{L^2(\mathbb{R}^d)}$ **with**

$$\mathcal{L}u(x) = \int_{|z|\geq 1} \frac{u(x+z)-u(x)}{|z|^{d+\beta}}[(1+|x|^2)^{q/2} + (1+|x+z|^2)^{q/2}]\,\mathrm{d}z$$

Extension of $\mathcal{L}$ **to** $C_b^2(\mathbb{R}^d)$ **by this equality**

▷ $\phi_\delta(x) := (1+|x|^2)^{-\delta/2}$ $(\delta > 0)$

$\Rightarrow M_t = \phi_\delta(X_t) - \phi_\delta(X_0) - \int_0^t \mathcal{L}\phi_\delta(X_s)\,\mathrm{d}s$: P_x**-martingale**

Proposition. $\forall q \in (\beta - d, \beta), \exists \delta > 0, \exists C > 0, \exists R > 0,$

$$\mathcal{L}\phi_\delta(x) \leq -\frac{C\phi_\delta(x)}{(1+|x|^2)^{(\beta-q)/2}} \leq 0 \quad (|x| \geq R).$$

▷ $B(r) = \{y \in \mathbb{R}^d \mid |y| \le r\}$: closed ball of radius $r > 0$

▷ $\sigma_r := \inf\{t > 0 \mid |X_t| \le r\}$: hitting time to $B(r)$

By the optional stopping theorem, if $|x| > R$, then

$$\text{(a)}\ E_x\left[\phi_\delta(X_{t\wedge\sigma_R})\right] = \phi_\delta(x) + E_x\left[\int_0^{t\wedge\sigma_R} \mathcal{L}\phi_\delta(X_s)\,\mathrm{d}s\right]$$

$$\le \phi_\delta(x),$$

$$\text{(b)}\ E_x\left[\phi_\delta(X_{t\wedge\sigma_R})\right] \ge \left\{\inf_{y\in B(R)} \phi_\delta(y)\right\} P_x(\sigma_R \le t)$$

$$\rightsquigarrow P_x(\sigma_R \le t) \le \frac{\phi_\delta(x)}{\inf_{y\in B(R)} \phi_\delta(y)} = \left(\frac{1+R^2}{1+|x|^2}\right)^{\delta/2} < 1$$

Hence $P_x(\sigma_R < \infty) < 1$ by letting $t \to \infty$.

By the change of variables formula, ($z = \sqrt{1+|x|^2}u$),

$$\mathcal{L}\phi_\delta(x) = \int_{|z|\geq 1} \frac{\phi_\delta(x+z) - \phi_\delta(x)}{|z|^{d+\beta}} [(1+|x|^2)^{q/2} + (1+|x+z|^2)^{q/2}] \,\mathrm{d}z$$
$$= \frac{\phi_\delta(x)(1+|x|^2)^{q/2}}{(1+|x|^2)^{\beta/2}} F_q(\delta, x) \quad (F_q(0,x) = 0)$$

Lemma. Assume that $d \geq 2$ and $q \in [\beta - d, 0)$.

(1) $\exists \delta_0 > 0, \forall \delta \in [0, \delta_0), \exists \lim_{|x|\to\infty} F_q(\delta, x) \ (=: \Lambda_q(\delta))$.

(2) $\Lambda_q(\delta)$ is continuously differentiable on $[0, \delta_0)$ and

$$\Lambda_q'(\delta) = -\int_0^\infty \left(\int_{-1}^1 \frac{\log(r^2+2rs+1)}{(r^2+2rs+1)^\delta} [1 + (r^2+2rs+1)^{q/2}](1-s^2)^{(d-3)/2} \,\mathrm{d}s \right) \frac{\mathrm{d}r}{r^{1+\beta}}.$$

(3) $q = \beta - d \Rightarrow \Lambda_q'(0) = 0$.

$$\Lambda_q'(0) = -\int_0^\infty \left(\int_{-1}^1 \log(r^2 + 2rs + 1)[1 + (r^2 + 2rs + 1)^{q/2}](1 - s^2)^{(d-3)/2} \, \mathrm{d}s \right) \frac{\mathrm{d}r}{r^{1+\beta}}$$

$\Rightarrow \forall q \in (\beta - d, 0), \Lambda_q'(0) < 0$ as $\Lambda_q'(0)$ is decreasing in q

- $\Lambda_q'(\delta)$ is continuous in δ by (2)
- $\Lambda_q(0) = 0$ by definition

$\Rightarrow \forall q \in (\beta - d, 0), \exists \delta_1 \in (0, \delta_0), \forall \delta \in (0, \delta_1),$ $\Lambda_q(\delta) < 0$

Hence $\forall \delta \in (0, \delta_1), \exists M > 0, \forall x \in \mathbb{R}^d$ $(|x| \geq M)$,

$$F_q(\delta, x) \leq \frac{\Lambda_q(\delta)}{2} < 0$$

This completes the proof of Proposition.

■ Related result (conservativeness/non-explosion)

(c) $\{T_t\}_{t>0}$ **is conservative** $\underset{\text{def}}{\iff}$ $T_t 1 = 1$, **a.e. for any** $t > 0$

▷ $\mathrm{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}\right)$: **symm. Hunt process on** $\mathbb{R}^d$

$$\Rightarrow T_t 1(x) = P_x\left(X_t \in \mathbb{R}^d\right)$$

▷ $\zeta := \inf\left\{t > 0 \mid X_t \notin \mathbb{R}^d\right\}$: **lifetime of** M

$\rightsquigarrow$ $\{T_t\}_{t>0}$ **is conservative** $\iff$ $P_x(\zeta = \infty) = 1$, **q.e.**

Theorem. **[S.-Uemura (14), S. (15)]**

$p \leq 2$ **(and** $q < \beta$**)** $\Longrightarrow$ $\{T_t\}_{t>0}$ **is conservative**