Compactness of semigroups generated by symmetric non-local Dirichlet forms with unbounded coefficients

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1. Introduction and result

▷ $\alpha \in (0,2)$, $p \in \mathbb{R}$, $q < \alpha$: fixed

$$\mathcal{E}(u,u) = \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|)^p + (1+|y|)^p]\,\mathrm{d}x\mathrm{d}y + \iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|)^q + (1+|y|)^q]\,\mathrm{d}x\mathrm{d}y$$

∘ $p = q = 0 \Rightarrow$ symmetric α-stable type

▷ $C_0^\infty(\mathbb{R}^d)$: totality of smooth functions on $\mathbb{R}^d$ with cpt. supp.

$\underset{[q<\alpha]}{\rightsquigarrow}$ $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$

$$\mathcal{E}(u,u) = \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|)^{p} + (1+|y|)^{p}]\,\mathrm{d}x\mathrm{d}y$$
$$+ \iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|)^{q} + (1+|y|)^{q}]\,\mathrm{d}x\mathrm{d}y$$

▷ $(\mathcal{E},\mathcal{F})$: closure of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d)$

($\rightsquigarrow$ regular Dirichlet form on $L^2(\mathbb{R}^d)$)

▷ $\{P_t\}_{t\geq 0}$: corresponding Markovian semigroup on $L^2(\mathbb{R}^d)$

Theorem. [S.-J. Wang, to appear in Potential Anal.]

Under the setting above,

$$\boxed{\exists/\forall t>0,\ P_t \text{ is compact on } L^2(\mathbb{R}^d) \iff p>2}$$

2. Comment on the proof of Theorem

P_t is compact $\underset{\text{[if } P_t \text{ has density]}}{\Longleftrightarrow}$ Super Poincaré inequality

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Super Poincaré inequality [F.-Y. Wang, 2002, 2005]

$$\begin{cases} \exists/\forall \psi \in L^2(\mathbb{R}^d) \ (\psi > 0), \\ \exists \beta : (0,\infty) \to (0,\infty)\text{: decreasing,} \end{cases}$$

$\forall u \in \mathcal{F}$, $\forall r > 0$,

$$\int_{\mathbb{R}^d} u^2 \, \mathrm{d}x \le r\mathcal{E}(u,u) + \beta(r) \left(\int_{\mathbb{R}^d} |u|\psi \, \mathrm{d}x \right) \cdots (*)$$

$\circ \displaystyle\int_{|x| \ge R_0} u^2 \, \mathrm{d}x \le \cdots$

$p > 2 \Rightarrow \forall \lambda > 0,\ \exists V_\lambda \in C^1_{++}(\mathbb{R}^d),\ \exists R_0 > 0,$

- $(1+|x|)^p + (1+|y|)^p \geq V_\lambda(x) + V_\lambda(y) \quad (|x-y| < 1)$
- $V_\lambda(x) = \lambda(1+|x|)^2 \quad (|x| \geq R_0)$

Hence

$$\mathcal{E}(u,u) \geq \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} [V_\lambda(x) + V_\lambda(y)]\, \mathrm{d}x\mathrm{d}y$$

$$=: \mathcal{E}^\lambda(u,u)$$

▷ $\phi(x) := (1+|x|^2)^{-\delta/2} \ (\delta \in (0,1))$

$$\mathcal{E}^\lambda(u,u) \geq \mathcal{E}^\lambda(\phi, u^2/\phi) = 2\int_{\mathbb{R}^d} \frac{-\mathcal{L}_\lambda \phi}{\phi} u^2\, \mathrm{d}x \quad \text{(Hardy-type)}$$

$$\mathcal{L}_\lambda \phi(x)$$

$$= \int_{0<|z|<1} (\phi(x+z) - \phi(x) - \langle \nabla\phi(x), z\rangle) \frac{V_\lambda(x) + V_\lambda(x+z)}{|z|^{d+\alpha}}\, dz$$

$$+ \frac{1}{2}\int_{0<|z|<1} \langle \nabla\phi(x), z\rangle \frac{V_\lambda(x+z) - V_\lambda(x-z)}{|z|^{d+\alpha}}\, dz$$

Calculation similar to [S.-Uemura, 2014]

Proposition. $\forall C > 0,\ \exists \lambda > 0,\ \exists R_0 > 0,\ \exists C_0 > 0,$

$$\boxed{\frac{-\mathcal{L}_\lambda \phi}{\phi}(x) \geq C 1_{\{|x| \geq R_0\}}(x) - C_0 1_{\{|x|<R_0\}}(x)\ (\forall x \in \mathbb{R}^d)}$$

$$\mathcal{E}(u,u) \geq 2\int_{\mathbb{R}^d} \frac{-\mathcal{L}_\lambda \phi}{\phi} u^2\, dx$$

$$\underset{\text{Prop.}}{\geq} 2C \underline{\int_{|x|\geq R_0} u^2\, dx} - 2C_0 \int_{|x|<R_0} u^2\, dx$$

Namely,

$$\int_{|x|\geq R_0} u^2\,\mathrm{d}x \leq \frac{1}{2C}\mathcal{E}(u,u) + \frac{C_0}{C}\boxed{\int_{|x|<R_0} u^2\,\mathrm{d}x}$$

Upper bound of $\boxed{\cdots}$ [Chen-Kim-Kumagai, 2008]

Nash inequality for the truncated symmetric stable process

($\rightsquigarrow \beta(r)$ comes from the heat kernel upper bound)

We finally take $r = C^{-1}$ so that

$$\int_{\mathbb{R}^d} u^2\,\mathrm{d}x \leq r\mathcal{E}(u,u) + \beta(r)\left(\int_{\mathbb{R}^d} |u|\psi\,\mathrm{d}x\right)\cdots(*)$$

for any $\psi \in L^2(\mathbb{R}^d)\cap\mathcal{B}_b(\mathbb{R}^d)$ $(\psi > 0)$

3. Related results

$$\mathcal{E}(u,u) = \iint_{|x-y|<1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|)^p + (1+|y|)^p]\,\mathrm{d}x\mathrm{d}y$$
$$+ \iint_{|x-y|\geq 1} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}[(1+|x|)^q + (1+|y|)^q]\,\mathrm{d}x\mathrm{d}y \quad (q<\alpha)$$

- **[Theorem]** $p \leq 2 \iff P_t$ **is non-compact on** $L^2(\mathbb{R}^d)$
- **[S.-Uemura, 2014; S., 2015]**

  $p \leq 2 \implies \{P_t\}_{t\geq 0}$ **is conservative**
- **[Okamura-Uemura, 2021; S., arXiv]**

  $p \leq 2-d$ **and** $q \leq \alpha - d \iff \{P_t\}_{t\geq 0}$ **is recurrent**