Maximal displacement of branching symmetric stable processes

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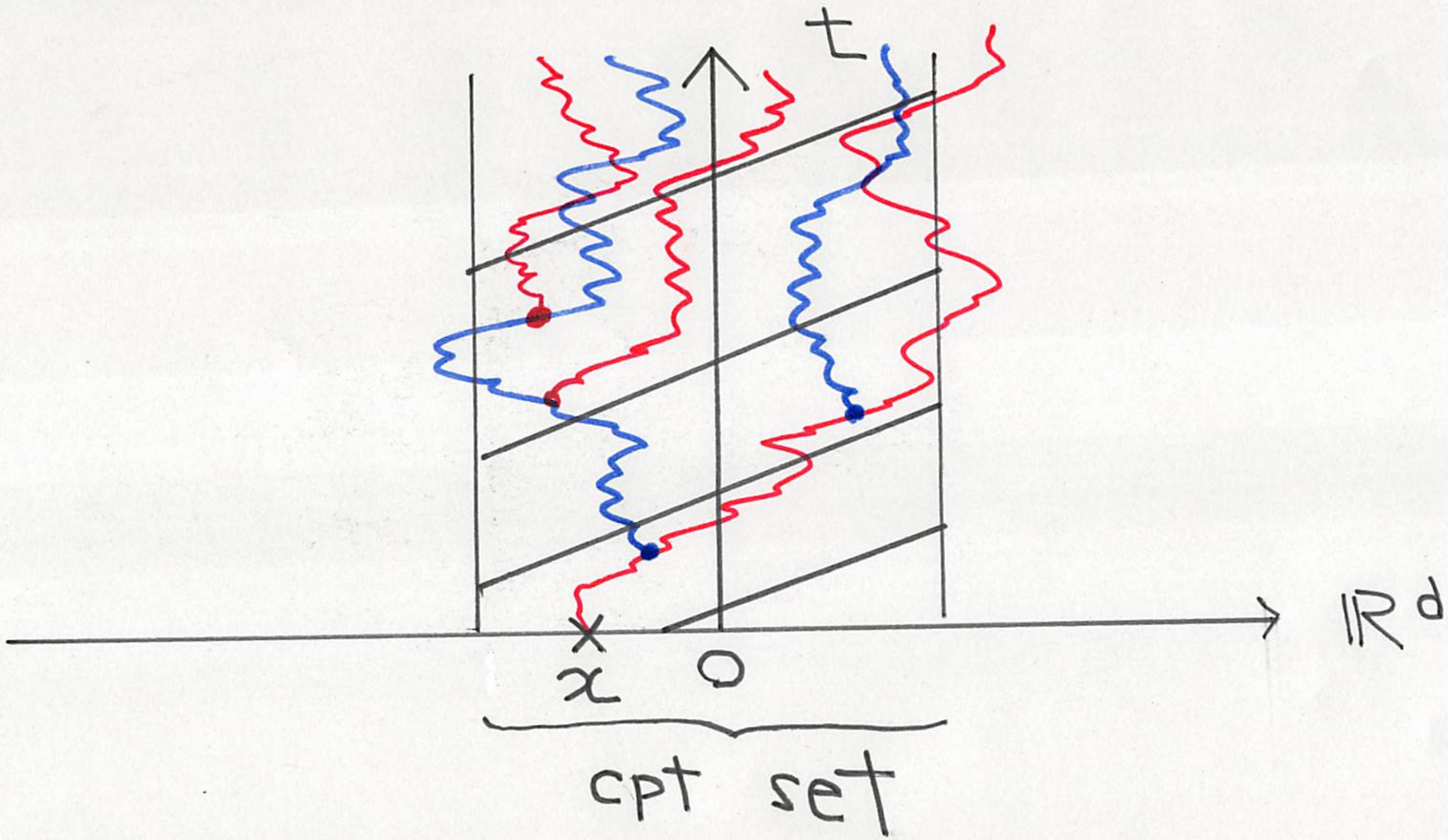
1. Introduction



○ Reproduction only on a compact set in $\mathbb{R}^d$

▷ L_t: forefront of the particle range at time t

▷ $(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$: **symm. α-stable proc. on $\mathbb{R}^d$**

generated by $-\frac{1}{2}(-\Delta)^{\alpha/2}$ $(\alpha \in (0,2))$

$$\Rightarrow \quad P_x(|X_t| > R) \sim \frac{c_0 t}{R^\alpha} \ (R \to \infty) \quad \text{(heavy tail)}$$

∘ **Branching RW on $\mathbb{Z}$ with spatially homogeneous branching**

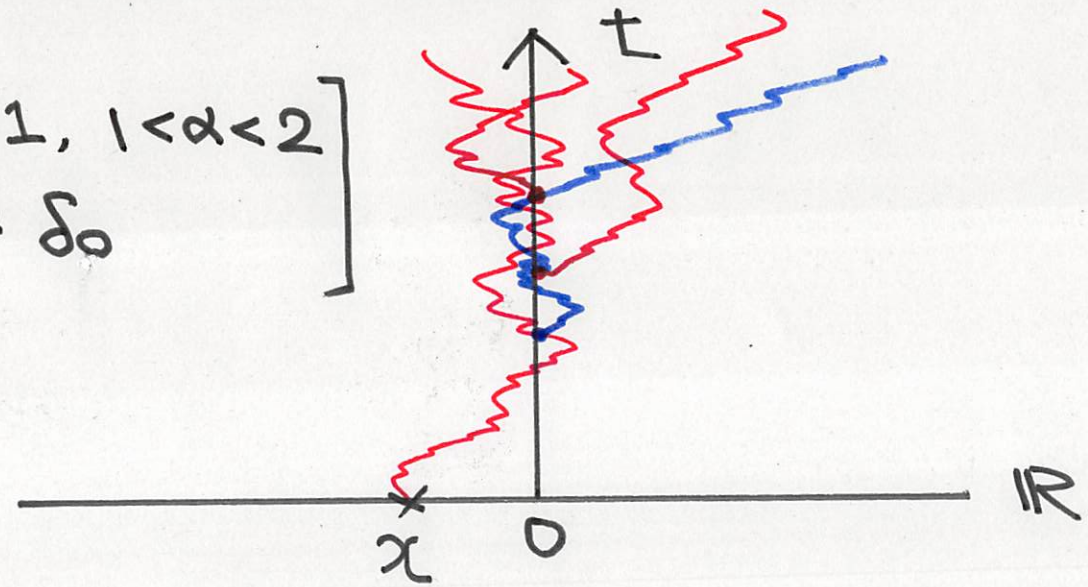
[Durrett(83), Bhattacharya-Hazra-Roy(17)]

∘ **(conti. time) Catalytic branching RW on $\mathbb{Z}$ [Bulinskaya(21)]**

(reproduction only on finite points)

2. Model and results

$d = 1, \ 1 < \alpha < 2$
$\mu = \delta_0$



(1) μ-killed symm. stable proc.
(2) binary branching at the lifetime
(3) indep. reproduction

▷ μ: positive Radon meas. on $\mathbb{R}^d$ with compact support

▷ $G_\beta(x, y)$: β-resolvent of the symm. α-stable proc. on $\mathbb{R}^d$

$$\lim_{\beta \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x, y)\, \mu(\mathrm{d}y) = 0 \quad \text{(Kato class)}$$

▷ $\lambda := \inf \mathrm{Spec}\left(\frac{1}{2}(-\Delta)^{\alpha/2} - \mu\right)$: **intensity of branching**

In what follows, we assume $\underline{\lambda < 0}$

$\Rightarrow$ **the ground state** $h \in C_b^+(\mathbb{R}^d)$ **exists and**

$$\boxed{h(x) \sim \frac{C_0}{|x|^{d+\alpha}} \int_{\mathbb{R}^d} h(y)\,\mu(\mathrm{d}y) \quad (|x| \to \infty)}$$

▷ $Z_t :=$ **population at time** t

▷ X_t^k: **position of the** k**th particle at time** t $(1 \le k \le Z_t)$

▷ $L_t := \max_{1 \le k \le Z_t} |\mathrm{X}_t^k|$:

maximal norm of particles alive at time t (**forefront**)

▷ $M_t := e^{\lambda t} \sum_{k=1}^{Z_t} h(X_t^k)$: nonneg. square integrable martingale

Theorem. $\exists c_* > 0$ (explicit), $\forall \kappa > 0$,

$$\lim_{t\to\infty} \mathbb{P}_x\left(e^{\lambda t/\alpha} L_t \leq \kappa\right) = \mathbb{E}_x\left[\exp\left(-\kappa^{-\alpha} c_* M_\infty\right)\right]$$

RHS: average over the Fréchet distribution with parameter α scaled by $c_* M_\infty$ [Bovier(17), Thm 1.12]

Remark. [(Non)degeneracy of M_∞]

- $d = 1$, $\alpha \in (1,2) \Rightarrow \lambda < 0$ and $\mathbb{P}_x(M_\infty > 0) = 1$
- $d > \alpha \Rightarrow \mathbb{P}_x(M_\infty = 0) \in (0,1)$

3. Comment on the proof of Theorem

▷ $R^{\kappa}(t) = \kappa e^{-\lambda t/\alpha}$ ($\kappa > 0$: fixed)

By the Markov and branching properties at time $T(\leq t)$,

$$\mathbb{P}_x(L_t \leq R^{\kappa}(t)) = \mathbb{E}_x\left[\prod_{k=1}^{Z_T} \mathbb{P}_{\mathrm{X}_T^k}(L_{t-T} \leq R^{\kappa}(t))\right] \cdots \text{(A)}$$

By the second moment method [Nishimori-S(21+)],

$$\mathbb{P}_{\mathrm{X}_T^k}(L_{t-T} \leq R^{\kappa}(t)) \asymp \exp\left(-\frac{c_*}{\kappa^{\alpha}} e^{\lambda T} h(\mathrm{X}_T^k)\right) \cdots \text{(B)}$$

By (A) and (B), we have as $t \to \infty$ and $T \to \infty$,

$$\mathbb{P}_x(L_t \leq R^{\kappa}(t)) \asymp \mathbb{E}_x\left[\exp\left(-\frac{c_*}{\kappa^{\alpha}} M_{\infty}\right)\right] \cdots \text{(C)}$$

4. Tail probability and examples

▷ $a(t)$: positive m'ble funct. s.t. $a(t) \to \infty$ $(t \to \infty)$

Theorem B. $\exists c_* > 0$ (as in Thm), loc. uniformly in $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\left(e^{\lambda t/\alpha} L_t > a(t)\right) \sim \frac{c_*}{a(t)^\alpha} h(x) \ (t \to \infty)$$

Example. [Catalytic branching] ▷ $d = 1$, $\alpha \in (1, 2)$

▷ $\mu = c\delta_0$ ($c > 0$, δ_0: Dirac meas. at the origin)

$\Rightarrow$ reproduction only at the origin

Theorems hold, λ and $h(x)$ can be written explicitly [S(08)].