Limiting distributions for the maximal displacement of branching Brownian motions

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1. Introduction

Branching Brownian motion on $\mathbb{R}^d$

- **Splitting time distribution**

$$\mathbb{P}_x(t < T \mid B_s, s \geq 0) = \exp\left(-\int_0^t V(B_s)\,\mathrm{d}s\right)$$

  - $\{B_t\}_{t\geq 0}$: trajectory of the initial Brownian particle
  - V: bounded nonnegative Borel function on $\mathbb{R}^d$

- **Offspring distribution** $\{p_n(x)\}_{n=1}^{\infty}$

$\rightsquigarrow$ **interaction between population growth and spatial motions**

Characterizations of the interaction

- **Asymptotic distribution of the population on a set**
- **Upper bound of the particle range (forefront)**

Spatially homogeneous model $(p_n(x) \equiv p_n,\ V(x) \equiv c)$

Bramson(78, 83), Mallein(15),...

Assume $p_2 = 1$ (binary branching) and $c = 1$

▷ R_t: **maximal norm of particles alive at time t (forefront)**

$$R_t = \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t + Y_t \ (t \to \infty)$$

Spatially inhomogeneous model

▷ $Q(x)=\sum_{n=1}^{\infty} np_n(x)$: expected offspring number at $x\in\mathbb{R}^d$

▷ $\mathcal{H}:=-\frac{1}{2}\Delta-(Q-1)V$: Schrödinger type operator

▷ $\lambda:=\inf\sigma(\mathcal{H})$: the bottom of the spectrum for $\mathcal{H}$

Assume V is small at infinity and $\lambda<0 \Rightarrow$

$R_t \sim \sqrt{\frac{-\lambda}{2}}t \ (t\to\infty)$ on the regular growth event

Erickson(84), Kolarov-Molchanov(13), Bocharov-Harris(14), S(18, 18+)

Limiting distributions of R_t

(i) Second order of R_t

(ii) Tail probability of R_t

(i) Lalley-Sellke(88)

Assume $d = 1$, $V \in C^+(\mathbb{R})$, $V(x) \to 0$ $(|x| \to \infty)$

$$\boxed{R_t = \sqrt{\frac{-\lambda}{2}}t + Y_t \quad (t \to \infty)}$$

$*$ **Bocharov-Harris(16): $d = 1$, $V = \delta_0$ (catalytic BBM)**

Purpose 1: To discuss the same problem for $d = 2$/singular V

(ii) Chauvin-Rouault(88,90) [Spatially homogeneous model]

Assume $p_2 = 1$ (binary branching) and $c = 1$

$\Rightarrow \forall \delta \geq \sqrt{2}$ and $\forall \kappa \in \mathbb{R}$, we have as $t \to \infty$,

$$\mathbb{P}_0(R_t > \delta t + \kappa) \sim \begin{cases} \dfrac{C_1}{t^{1/2}} \exp\left(\dfrac{1}{2}(\delta^2 - 2)t - \delta\kappa\right) & (\delta > \sqrt{2}) \\ \dfrac{C_2 \log t}{t^{3/2}} e^{-\sqrt{2}\kappa} & (\delta = \sqrt{2}) \end{cases}$$

Purpose 2: To find the tail probability asymptotics

for the spatially inhomogeneous model

2. Model and results

▷ $G_\alpha(x,y)$: α-resolvent of d-dim BM.

▷ μ: positive Radon measure on $\mathbb{R}^d$

$$\mu \in \mathcal{K} \underset{\text{def}}{\iff} \lim_{\alpha\to\infty} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} G_\alpha(x,y)\,\mu(\mathrm{d}y) = 0$$

- **Splitting time distribution**

$$\mathbb{P}_x(t < T \mid B_s, s \geq 0) = e^{-A_t^\mu}$$

A_t^μ: positive conti. additive f'nal $\leftrightarrow$ μ (Revuz corresp.)

- **Offspring distribution** $\sim \{p_n(x)\}_{n=1}^\infty$ (prob. funct. on $\mathbb{R}^d$)

$$\mathbb{P}_x(t < T \mid B_s, s \geq 0) = e^{-A_t^\mu}$$

Revuz correspondence

For any $f, h \in \mathcal{B}^+(\mathbb{R}^d)$,

$$\lim_{t \to 0} \frac{1}{t} E_{hm}\left[\int_0^t f(B_s)\,\mathrm{d}A_s^\mu\right] = \int_{\mathbb{R}^d} f(x)h(x)\,\mu(\mathrm{d}x)$$

($m(\mathrm{d}x) = \mathrm{d}x$: d-dim. Lebesgue measure)

Example.

(i) $\mu(\mathrm{d}x) = V(x)\,\mathrm{d}x \Rightarrow A_t^\mu = \int_0^t V(B_s)\,\mathrm{d}s$

(ii) $d = 1$, $\mu = \delta_0 \Rightarrow A_t^{\delta_0} = 2l_t$ (l_t: local time at $x = 0$)

Forefront of particles

- ▷ $Z_t :=$ population at time t
- ▷ B_t^k: position of the kth particle at time t $(1 \leq k \leq Z_t)$
- ▷ $R_t := \max_{1 \leq k \leq Z_t} |\mathrm{B}_t^k|$:

  maximal norm of particles alive at time t (forefront)

Intensity of branching

- ▷ $Q(x) := \sum_{n=1}^{\infty} n p_n(x)$: expected offspring number at $x \in \mathbb{R}^d$
- ▷ $\lambda := \inf \sigma \left(-\frac{1}{2}\Delta - (Q-1)\mu \right)$: intensity of branching

▷ $R(x) := \sum_{n=1}^{\infty} n(n-1)p_n(x)$

Assumption.

(i) μ **has compact support and** $R\mu \in \mathcal{K}$ **(**$\Rightarrow$ $(Q-1)\mu \in \mathcal{K}$**)**

(ii) $\lambda < 0$

(i) $\Rightarrow$ **particles can branch only on a compact set**

(ii) $\Rightarrow$ **the intensity of branching is strong enough**

Analytic consequence of Assumption [Takeda(03, 08)]

$\exists$**ground state** $h \in C_b^+(\mathbb{R}^d)$, $h(x) \asymp G_{-\lambda}(0,x) \asymp \dfrac{e^{-\sqrt{-2\lambda}|x|}}{|x|^{(d-1)/2}}$

Result 1: Second order of R_t

$$R_t = \sqrt{\frac{-\lambda}{2}}t + \frac{d-1}{\sqrt{-2\lambda}}\log t + Y_t$$

- ▷ $Z_t(h) := \sum_{k=1}^{Z_t} h(\mathrm{B}_t^k)$: population at time t weighted by h
- ▷ $M_t = e^{\lambda t} Z_t(h)$ (normalization): nonnegative $\mathbb{P}_x$-martingale

Theorem 1. If $d = 1, 2$, then $\exists c_* > 0$ (explicit) s.t. $\forall \kappa \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{P}_x\left(Y_t \leq \kappa\right) = \mathbb{E}_x\left[\exp\left(-c_* e^{-\sqrt{-2\lambda}\kappa} M_\infty\right)\right]$$

(Gumbel type distribution appears)

Remark. **[S(18+)]**

- $d = 1, 2 \Rightarrow \mathbb{P}_x(M_\infty > 0) = 1$
- $d \geq 3 \Rightarrow \mathbb{P}_x(M_\infty = 0) > 0$ **and**

$$\limsup_{t \to \infty} \frac{R_t}{\sqrt{2t \log \log t}} = 1 \text{ on } \{M_\infty = 0\}$$

Hence Theorem 1 is not true as it is.

Result 2: Tail probability of R_t

$$\boxed{R_t \sim \sqrt{\frac{-\lambda}{2}}\, t \ (t \to \infty) \text{ on } \{M_\infty > 0\}}$$

Asymptotics of $\mathbb{P}_x(R_t > r(t))$ **as** $t \to \infty$

▷ Z_t^r = population on $\{y \in \mathbb{R}^d \mid |y| > r\}$ $(r > 0)$

Note. $\{R_t > r\} = \{Z_t^r \geq 1\}$

Theorem 2. **(1) (Subcritical case)**

▷ $a(t)$: nondecreasing function s.t. $a(t) = o(t)$ $(t \to \infty)$

▷ $r_1(t) := \delta t + a(t)$ $(\delta \in (\sqrt{-\lambda/2}, \sqrt{-2\lambda}))$

$\Rightarrow \forall K \subset \mathbb{R}^d$: compact

$$\lim_{t\to\infty} \inf_{x\in K} \frac{\mathbb{P}_x(R_t > r_1(t))}{\mathbb{E}_x[Z_t^{r_1(t)}]} = \lim_{t\to\infty} \sup_{x\in K} \frac{\mathbb{P}_x(R_t > r_1(t))}{\mathbb{E}_x[Z_t^{r_1(t)}]} = 1$$

(2) (Critical case)

▷ $b(t)$: nondecreasing function s.t. $b(t) = o(\log t)$ $(t \to \infty)$

▷ $r_2(t) := \sqrt{\frac{-\lambda}{2}}t + \frac{\gamma}{\sqrt{-2\lambda}} \log t + b(t)$ ($\gamma > d + 1$)

(Technical) assumption:

$\mu \ll m$ **and the density function is bounded**

$\Rightarrow \forall K \subset \mathbb{R}^d$**: compact**

$$\lim_{t\to\infty} \inf_{x\in K} \frac{\mathbb{P}_x(R_t > r_2(t))}{\mathbb{E}_x[Z_t^{r_2(t)}]} = \lim_{t\to\infty} \sup_{x\in K} \frac{\mathbb{P}_x(R_t > r_2(t))}{\mathbb{E}_x[Z_t^{r_2(t)}]} = 1$$

- $\delta \in (\sqrt{-\lambda/2}, \sqrt{-2\lambda})$

$$\mathbb{P}_x(R_t > \delta t) \sim \mathbb{E}_x\left[Z_t^{\delta t}\right] = E_x\left[e^{A_t^{(Q-1)\mu}}; |B_t| > \delta t\right]$$

$$\sim c_d \delta^{(d-1)/2} e^{(-\lambda - \sqrt{-2\lambda}\delta)t} t^{(d-1)/2} h(x)$$

- $\gamma > d+1$

$$\mathbb{P}_x\left(R_t > \sqrt{\frac{-\lambda}{2}}t + \frac{\gamma}{\sqrt{-2\lambda}}\log t\right)$$

$$\sim c_d \delta^{(d-1)/2} t^{(d-1-\gamma)/2} h(x)$$

Note. **[S(18+)]** $\forall \delta \geq \sqrt{-2\lambda}$,

$$\mathbb{P}_x(R_t > \delta t) \asymp P_x\left(|B_t| > \delta t\right) \asymp e^{-\delta^2 t/2} t^{(d-2)/2}$$

Result 3: Yaglom type limit

$$\mathbb{P}_x(R_t > r(t)) = \mathbb{P}_x\left(Z_t^{r(t)} \geq 1\right) \to 0 \ (t \to \infty)$$

Conditional distribution of $Z_t^{r(t)}$ on the event $\{Z_t^{r(t)} \geq 1\}$

Theorem 3. **Under the same setting as in Theorem 2,**

$$\lim_{t\to\infty} \mathbb{P}_x\left(Z_t^{r_j(t)} = k \mid Z_t^{r_j(t)} \geq 1\right) = \begin{cases} 1 & (k = 1) \\ 0 & (k \geq 2) \end{cases}$$

for $j = 1, 2$.

3. Sketch of the proofs

Proof of Theorem 1.

$$R_t = \sqrt{\frac{-\lambda}{2}} t + \frac{d-1}{\sqrt{-2\lambda}} \log t + Y_t$$

Theorem 1. If $d = 1, 2$, then $\exists c_* > 0$ (explicit) s.t. $\forall \kappa \in \mathbb{R}$,

$$\lim_{t \to \infty} \mathbb{P}_x \left(Y_t \leq \kappa \right) = \mathbb{E}_x \left[\exp \left(-c_* e^{-\sqrt{-2\lambda}\kappa} M_\infty \right) \right]$$

▷ $r(t) = \sqrt{\frac{-\lambda}{2}} t + \frac{d-1}{\sqrt{-2\lambda}} \log t + \kappa$

$$\mathbb{P}_x \left(Y_t \leq \kappa \right) = \mathbb{P}_x \left(R_t \leq r(t) \right)$$

∘ Follow the argument of Bocharov-Harris(16)

Step 1 (Conditioning on the initial points (1)).

▷ $s(t)$: $s(t) \to \infty$ and $s(t) = o(t)$ $(t \to \infty)$

$\boxed{R_t \sim \sqrt{-\lambda/2}t \ (t \to \infty)}$ and Markov property $\Rightarrow \forall \varepsilon > 0$,

$$\mathbb{P}_x \left(R_t \le r(t)\right)$$

$$\sim \mathbb{E}_x \left[\mathbb{P}_{\mathrm{B}_{s(t)}} \left(R_{t-s(t)} \le r(t)\right); R_{s(t)} \le \left(\sqrt{\frac{-\lambda}{2}} + \varepsilon\right) s(t)\right]$$

- $R_{s(t)} \le \left(\sqrt{\frac{-\lambda}{2}} + \varepsilon\right) s(t) \Rightarrow |\mathrm{B}^k_{s(t)}| \le \left(\sqrt{\frac{-\lambda}{2}} + \varepsilon\right) s(t)$
- $\mathbb{P}_{\mathrm{B}_{s(t)}} \left(R_{t-s(t)} \le r(t)\right) = \prod_{k=1}^{Z_t} \mathbb{P}_{\mathrm{B}^k_{s(t)}} \left(R_{t-s(t)} \le r(t)\right)$

Step 2 (Conditioning on the initial points (2)).

▷ σ_K: hitting time of some particle to $K := \mathrm{supp}[\mu]$

▷ $s(t) = 1 \vee \log t \ (d = 1) / s(t) = 1 \vee \log\log t \ (d = 2)$

▷ $\beta \in (0, 1/2)$: fixed

By the strong Markov property and tail estimate of σ_K

[Byczkowski-Małecki-Ryznar(13) for $d = 2$],

$$\mathbb{P}_{\mathrm{B}^k_{s(t)}}\left(R_{t-s(t)} \leq r(t)\right)$$

$$\fallingdotseq E_{\mathrm{B}^k_{s(t)}}\left[\boxed{\mathbb{P}_{B_{\sigma_K}}\left(R_{t-s(t)-s} \leq r(t)\right)|_{s=\sigma_K}} ; \sigma_K \leq \beta t\right]$$

$$\triangleright\ \eta(t) := e^{-\lambda t}\int_{|z|>r(t)} h(z)\,\mathrm{d}z \quad \left(\sim c_* e^{-\sqrt{-2\lambda}\kappa}\right)$$

$$\mathbb{P}_{\mathrm{B}^k_{s(t)}}\left(R_{t-s(t)} \le r(t)\right)$$

$$\fallingdotseq E_{\mathrm{B}^k_{s(t)}}\left[\boxed{\mathbb{P}_{B_{\sigma_K}}\left(R_{t-s(t)-s} \le r(t)\right)\big|_{s=\sigma_K}}\,;\ \sigma_K \le \beta t\right]$$

$$\gtrsim 1 - e^{\lambda s(t)} h(\mathrm{B}^k_{s(t)})\eta(t) \fallingdotseq \exp\left(-e^{\lambda s(t)} h(\mathrm{B}^k_{s(t)})\eta(t)\right)$$

$(\because 1 - t \fallingdotseq e^{-t}$ near $t = 0)$

$$\prod_{k=1}^{Z_t} \mathbb{P}_{\mathrm{B}^k_{s(t)}}\left(R_{t-s(t)} \le r(t)\right) \gtrsim \exp\left(-M_{s(t)}\eta(t)\right)$$

$$\to \exp\left(-c_* e^{-\sqrt{-2\lambda}\kappa} M_\infty\right)$$

Step 3 (Verification of the next inequality).

$$E_x\left[\boxed{\mathbb{P}_{B_{\sigma_K}}\left(R_{t-s(t)-s} \le r(t)\right)|_{s=\sigma_K}}; \sigma_K \le \beta t\right]$$

$$\gtrsim 1 - e^{\lambda s(t)} h(x)\eta(t)$$

$\forall y(= B_{\sigma_K}) \in K$ and $s(= \sigma_K) \in (0, \beta t]$,

$$\boxed{\mathbb{P}_y\left(R_{t-s(t)-s} \le r(t)\right)} \ge 1 - \mathbb{E}_y\left[Z^{r(t)}_{t-s(t)-s}\right]$$

$$\fallingdotseq 1 - e^{\lambda(s(t)+s)} h(y)\eta(t)$$

- Optional stopping thm for P_x-martingale $e^{\lambda t + A_t^{(Q-1)\mu}} h(B_t)$
- Tail estimate of σ_K

$$\mathbb{E}_y\left[Z^{r(t)}_{t-s(t)-s}\right] = E_y\left[e^{A^{(Q-1)\mu}_{t-s(t)-s}}; |B_{t-s(t)-s}| > r(t)\right]$$

$$\sim e^{\lambda(s(t)+s)}h(y)\eta(t)$$

▷ λ_2: the second bottom of the spectrum for $-\frac{1}{2}\Delta - (Q-1)\mu$

$\Rightarrow \lambda < \lambda_2 \leq 0$ [e.g., Ben Amor(04)]

Poincaré inequality [Chen-S(07)]

$\forall \varphi \in L^2(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \varphi h \, \mathrm{d}x = 0$,

$$\left| E_x\left[e^{A_t^{(Q-1)\mu}}\varphi(B_t)\right]\right| \leq Ce^{-\lambda_2 t}\|\varphi\|_{L^2(\mathbb{R})}$$

$\Rightarrow$ estimate of the Feynman-Kac semigroup

Proof of Theorem 2.

Theorem 2. **(Subcritical case)**

▷ $a(t)$: **nondecreasing function s.t.** $a(t) = o(t)$ $(t \to \infty)$

▷ $r_1(t) := \delta t + a(t)$ $\quad (\delta \in (\sqrt{-\lambda/2}, \sqrt{-2\lambda}))$

$\Rightarrow \forall K \subset \mathbb{R}^d$: **compact,**

$$\lim_{t\to\infty} \inf_{x\in K} \frac{\mathbb{P}_x(R_t > r_1(t))}{\mathbb{E}_x[Z_t^{r_1(t)}]} = \lim_{t\to\infty} \sup_{x\in K} \frac{\mathbb{P}_x(R_t > r_1(t))}{\mathbb{E}_x[Z_t^{r_1(t)}]} = 1$$

Upper bound.

$$\mathbb{P}_x(R_t > r_1(t)) = \mathbb{P}_x(Z_t^{r_1(t)} \geq 1) \leq \mathbb{E}_x\left[Z_t^{r_1(t)}\right]$$

Lower bound (Feynman-Kac expression).

McKean(75): the spatially homogeneous model

$$u_r(t,x) := \mathbb{P}_x(R_t \le r)$$

$$= E_x\left[e^{-A_t^\mu}; |B_t| \le r\right]$$

$$+ E_x\left[\int_0^t e^{-A_s^\mu} \sum_{n=1}^{\infty} p_n(B_s) u_r(t-s, B_s)^{(n-1)+1} \, \mathrm{d}A_s^\mu\right]$$

Formally,

$$\frac{\partial u_r}{\partial t} = \left[\frac{1}{2}\Delta + \mu\left(\sum_{n=1}^{\infty} p_n u_r^{n-1} - 1\right)\right] u_r$$

▷ $v_r(t,x) := 1 - u_r(t,x) = \mathbb{P}_x(R_t > r)$

$$\mathbb{P}_x(R_t > r_1(t)) = v_{r_1(t)}(t,x) = E_x\left[e^{C_t}; |B_t| > r_1(t)\right]$$

$*$ **Lower bound of the RHS by the next inequality:**

$$\mathbb{P}_x(R_t > r_1(t)) \le \mathbb{E}_x\left[Z_t^{r_1(t)}\right] = E_x\left[e^{A_t^{(Q-1)\mu}}; |B_t| > r_1(t)\right]$$

$$\sim e^{-\lambda t} h(x) \int_{|z|>r_1(t)} h(z)\,\mathrm{d}z$$

Critical case $(r_2(t) := \sqrt{-\lambda/2}t + \cdots)$.

Assumption: $\mu \ll m$ **and** $V := \dfrac{\mathrm{d}\mu}{\mathrm{d}m}$ **is bounded**

$\Rightarrow A_t^{\mu} = \int_0^t V(B_s)\,\mathrm{d}s \lesssim t$ **and**

$$E_x\left[e^{A_t^{\mu}} A_t^{\mu}; |B_t| > r_2(t)\right] \lesssim t E_x\left[e^{A_t^{\mu}}; |B_t| > r_2(t)\right] \le \cdots$$