Spread rate of branching Brownian motions

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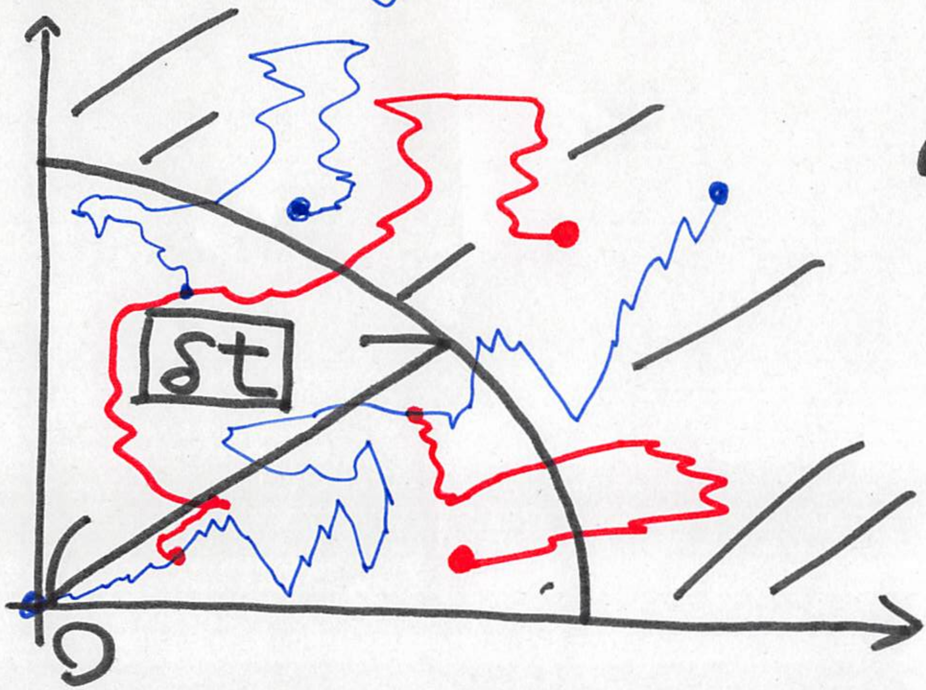
1. Introduction

- Branching Brownian motions.



$\mathbb{R}^d$

at time t

$Z_t^{\delta t}$: population on ////

($\delta \geq 0$: const.)

Spatially inhomogeneous

- Splitting time dist. ... Trajectory dep.
- Offspring dist. ... branching site dep.

▷ λ: principal eigenvalue of some Schrödinger type operator

($-\lambda$: intensity of branching)

Assume $\underline{\lambda < 0}$

◦ $\delta > \sqrt{-\lambda/2} \Rightarrow Z_t^{\delta t} = 0$ eventually a.s.

◦ $\delta < \sqrt{-\lambda/2} \Rightarrow \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\delta t} = -\lambda - \sqrt{-2\lambda}\delta \ (> 0)$

on the regular growth event

[Koralov-Molchanov(13), Bocharov-Harris(14), S(18)]

▷ R_t: **maximal displacement at time** t

⇒ **On the regular growth event,**

$$R_t \sim \sqrt{-\lambda/2}\, t \quad (t \to \infty)$$

[Erickson(84), Koralov-Molchanov(13), BH(14), S(18)]

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x(R_t \geq \delta t) = \begin{cases} -\delta^2/2 & (\delta \geq \sqrt{-2\lambda}) \\ -\lambda - \sqrt{-2\lambda}\delta & (\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda}) \end{cases}$$

Purpose in this talk.

(i) Growth rate of $Z_t^{\delta t}$ at $\delta = \sqrt{-\lambda/2}$

(ii) More precise asymptotics of $\mathbb{P}_x(R_t \geq \delta t)$

(iii) Growth rates of $Z_t^{\delta t}$ and R_t on the survival event

Note. Spatially homogeneous case:

- Growth of $Z_t^{\delta t}$: Biggins(95, 96)
- Growth of R_t: Bramson(79), Roberts(13), Kyprianou(05)
- Upper deviation for R_t: Chauvin-Rouault(88)

2. Model and results

* **Splitting time distribution**

$$P_x(t < T \mid B_s, s \geq 0) = \exp\left(-A_t^{\mu}\right)$$

* **Offspring distribution** $\sim \{p_n(x)\}_{n=1}^{\infty}$

○ A_t^{μ}: **positive continuous additive functional**

* $\mu(\mathrm{d}x) = V(x)\,\mathrm{d}x \Rightarrow A_t^{\mu} = \int_0^t V(B_s)\,\mathrm{d}s$
* $\mu = \delta_0 \Rightarrow A_t^{\delta_0} = 2l_t$ (l_t: **local time at** $x = 0$)

▷ $Q(x) := \sum_{n=1}^{\infty} n p_n(x)$

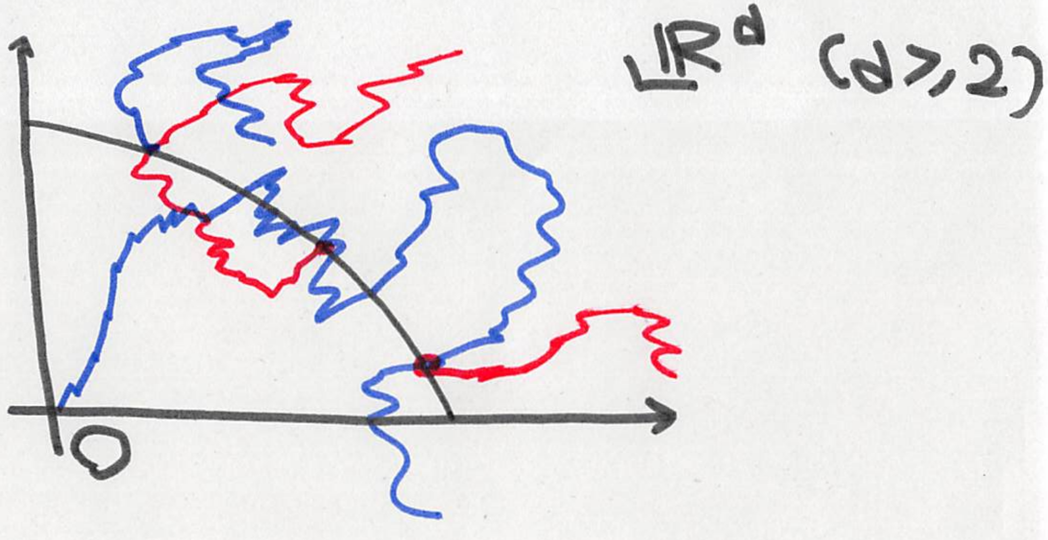
▷ $\nu(\mathrm{d}x) := (Q(x) - 1)\,\mu(\mathrm{d}x)$: intensity of branching

Assumption. ν belongs to a certain Kato class

Example. (i) $0 \le V(x) \le |x|^{-p} 1_{\{0<|x|\le R\}}$

* $\underline{d = 1 \text{ and } p < 1}$ * $\underline{d \ge 2 \text{ and } p < 2}$

(ii) Dirac/surface measure



▷ Z_t := population at time t

▷ B_t^k: position of the kth particle at time t $(1 \le k \le Z_t)$

▷ $Z_t(f) := \sum_{k=1}^{Z_t} f(\mathrm{B}_t^k)$, $Z_t(A) := Z_t(1_A)$ $(A \subset \mathbb{R}^d)$

$$\Rightarrow \mathbb{E}_x[Z_t(f)] = E_x\left[e^{A_t^\nu} f(B_t)\right] = e^{(\Delta/2+\nu)t} f(x)$$

▷ $\lambda := \inf \sigma\left(-\frac{1}{2}\Delta - \nu\right)$

∗ $\boxed{\lambda < 0}$ $\Rightarrow$ ∃ground state h [Takeda ('03)]

$$\mathbb{E}_x[Z_t(h)] = e^{(\Delta/2+\nu)t} h(x) = e^{-\lambda t} h(x)$$

▷ $M_t := e^{\lambda t} Z_t(h)$: **nonnegative $\mathbb{P}_x$-martingale**

▷ $M_\infty := \lim_{t\to\infty} M_t$

Limit theorem: $\forall A \subset \mathbb{R}^d$: **rel. cpt. open with** $|\partial A| = 0$,

$$Z_t(A) \sim e^{-\lambda t} \int_A h(y)\,\mathrm{d}y\, M_\infty \quad (t \to \infty)$$

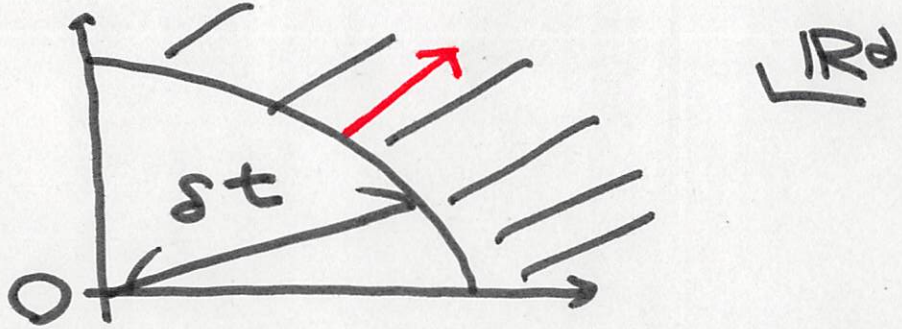
[S. Watanabe(67), Asmussen-Hering(76),

Z.-Q. Chen-S.(07), Engländer-Harris-Kyprianou(10),

Z.-Q. Chen-Y.-X. Ren-T. Yang(17)]

○ **Koralov-Molchanov (13)**: $Z_t(A + tv)$ $(|v| < \sqrt{-\lambda/2})$

▷ $Z_t^{\delta t} := Z_t(\{|x| \geq \delta t\})$



Assumption.

(i) ν belongs to a certain Kato class and has cpt. supp.

(ii) $\lambda < 0$ and $\mathbb{P}_x(M_\infty > 0) > 0$ ($\Leftarrow$ regular growth)

▷ $\mathbb{P}_x^*(\cdot) := \mathbb{P}_x(\cdot \mid M_\infty > 0)$

Theorem 1. $\delta := \sqrt{-\lambda/2}$, $d \geq 3$

$\Rightarrow$ $Z_t^{\delta t}$ grows polynomially $\mathbb{P}_x^*$-a.s.:

$$\frac{d-2}{2} \leq \liminf_{t\to\infty} \frac{\log Z_t^{\delta t}}{\log t} \leq \limsup_{t\to\infty} \frac{\log Z_t^{\delta t}}{\log t} \leq \frac{d+3}{2}$$

Formal observation.

Since $h(x) \asymp e^{-\sqrt{-2\lambda}|x|}/|x|^{(d-1)/2}$ $(|x| \geq 1)$,

$$Z_t^{\delta t} \overset{??}{\sim} e^{-\lambda t} \int_{|y|\geq \delta t} h(y)\,\mathrm{d}y\, M_\infty \asymp e^{(-\lambda-\sqrt{-2\lambda}\delta)t} t^{(d-1)/2}$$

$$\boxed{Z_t^{\sqrt{-\lambda/2}t} \overset{??}{\asymp} t^{(d-1)/2}}$$

Upper bound. $\exists \{t_n\}$, $\exists G(t) \nearrow \infty$ s.t.

$$\sum_{n=1}^{\infty} \mathbb{P}_x \left(\max_{t_n \leq s \leq t_{n+1}} Z_s^{\delta t_n} \geq G(t_n) \right) < \infty$$

$\Rightarrow$ by the Borel-Cantelli lemma, $\forall n$: large, $\forall t \in [t_n, t_{n+1}]$,

$$Z_t^{\delta t} \leq \max_{t_n \leq s \leq t_{n+1}} Z_s^{\delta t_n} \leq G(t_n) \leq G(t)$$

▷ $R_t := \max_{1\le k\le Z_t} |\mathrm{B}_t^k|$: **maximal displacement**

Theorem 2. **Suppose** $\sup_{x\in\mathbb{R}^d}\sum_{n=1}^{\infty} n^2 p_n(x) < \infty$.

(i) $\delta \ge \sqrt{-2\lambda} \Rightarrow$ $\boxed{\mathbb{P}_x(R_t \ge \delta t) \asymp e^{-\delta^2 t/2} t^{(d-2)/2}}$

(ii) $\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda} \Rightarrow \exists c_1, c_2 > 0$ **s.t.** $\forall t$: **large,**

$$c_1 e^{(-\lambda-\sqrt{-2\lambda}\delta)t} t^{(d-2)/2} \le \mathbb{P}_x(R_t \ge \delta t) \le c_2 e^{(-\lambda-\sqrt{-2\lambda}\delta)t} t^{(d-1)/2}$$

∘ $\delta \ge \sqrt{-2\lambda} \Rightarrow$ $\boxed{\mathbb{P}_x(R_t \ge \delta t) = \mathbb{P}_x(Z_t^{\delta t} \ge 1) \asymp \mathbb{E}_x[Z_t^{\delta t}]}$

Feynman-Kac expression **[McKean(76), Chauvin-Rouault(88)]**

Remark.

- $\sup_{x\in\mathbb{R}^d}\sum_{n=1}^{\infty}(n\log n)p_n(x)<\infty\Rightarrow\mathbb{P}_x(M_\infty>0)>0$

  ($L\log L$ **condition**) **[Z.-Q. Chen-Y.-X. Ren-T. Yang(17)]**

- $\underline{d=1,2}$ **and** $\underline{L\log L \text{ condition}}$

  $\Rightarrow \lambda<0$ **[Takeda(03)] and** $\mathbb{P}_x(M_\infty>0)=1$ **[S(08)]**

- $d\geq 3\Rightarrow\mathbb{P}_x(M_\infty=0)>0$ **and**

$$\limsup_{t\to\infty}\frac{R_t}{\sqrt{2t\log\log t}}=1,\quad \mathbb{P}_x(\cdot|M_\infty=0)\text{-a.s.}$$

3. Maximal displacement/population growth on the survival

Assumption.

(i) ν is small enough at infinity.

(ii) $\lambda < 0$ and $\mathbb{P}_x(M_\infty > 0) > 0$

(iii) $\boxed{p_0 \not\equiv 0}$

Theorem 3. Under Assumption,

- $\delta > \sqrt{-\lambda/2} \Rightarrow Z_t^{\delta t} = 0$ eventually a.s.
- $\delta < \sqrt{-\lambda/2} \Rightarrow \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\delta t} = -\lambda - \sqrt{-2\lambda}\delta$, $\mathbb{P}_x^*$-a.s.

▷ $e_0 := \inf\{t > 0 \mid Z_t = 0\}$: **extinction time**

▷ $R_t = \left(\max_{1 \le k \le Z_t} |\mathrm{B}_t^k|\right) 1_{\{t < e_0\}}$

Corollary. **Under Assumption,** $\boxed{R_t \sim \sqrt{-\lambda/2}\,t}$ $\mathbb{P}_x^*$**-a.s.**

Remark. $d = 1, 2 \Rightarrow$ **We can replace** $\mathbb{P}_x^*$ **by** $\mathbb{P}_x(\cdot \mid e_0 = \infty)$

Theorem 4. **Under Assumption,**

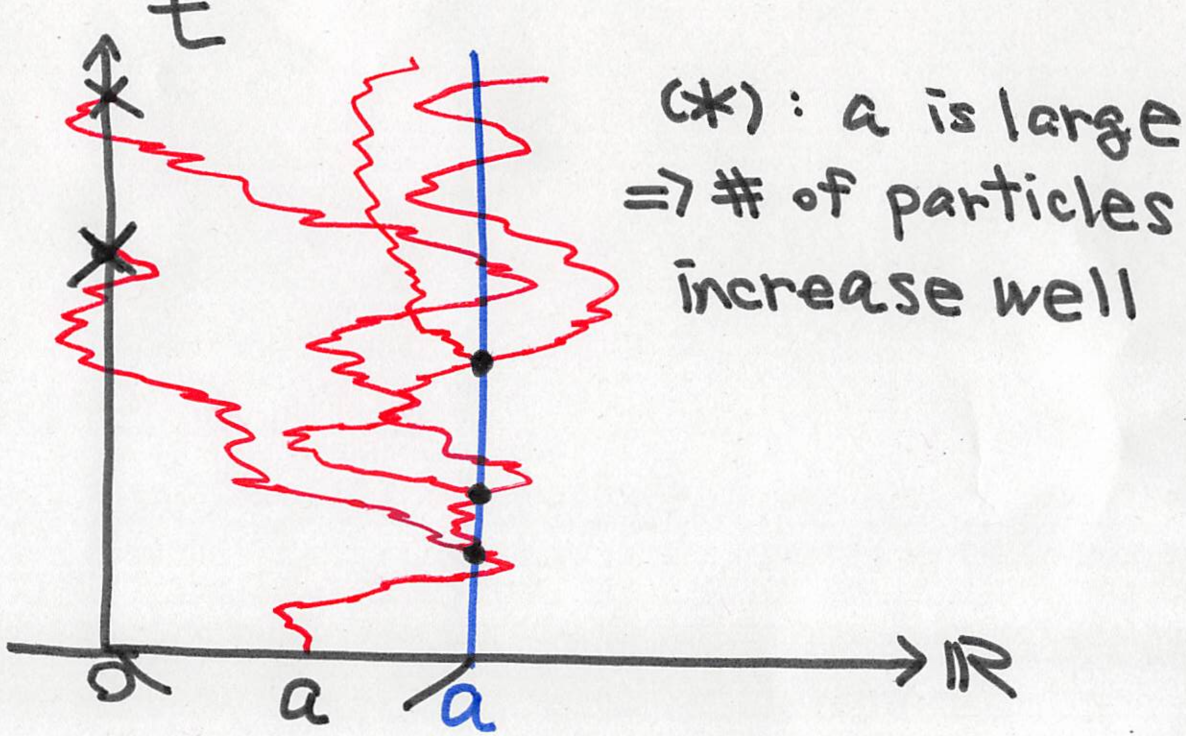
$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x(R_t \ge \delta t) = \begin{cases} -\delta^2/2 & (\delta \ge \sqrt{-2\lambda}) \\ -\lambda - \sqrt{-2\lambda}\,\delta & (\sqrt{-\lambda/2} < \delta < \sqrt{-2\lambda}) \end{cases}$$

Example.

▷ $\mu = \delta_0 + \delta_a \quad (a > 0)$

▷ $p_0(x) + p_2(x) \equiv 1$

▷ $p_0(0) = 1, \; q = p_2(a)$



According to [Takeda(02)],

$$\lambda((Q-1)\mu) < 0 \iff q > \frac{1}{2}\left(1 + \frac{1}{1+2a}\right) \qquad (*)$$

$(*) \Rightarrow$ **Theorem 3 and Corollary hold** $\mathbb{P}_x(\cdot \mid e_0 = \infty)$**-a.s.**