Spread rate of branching Brownian motions

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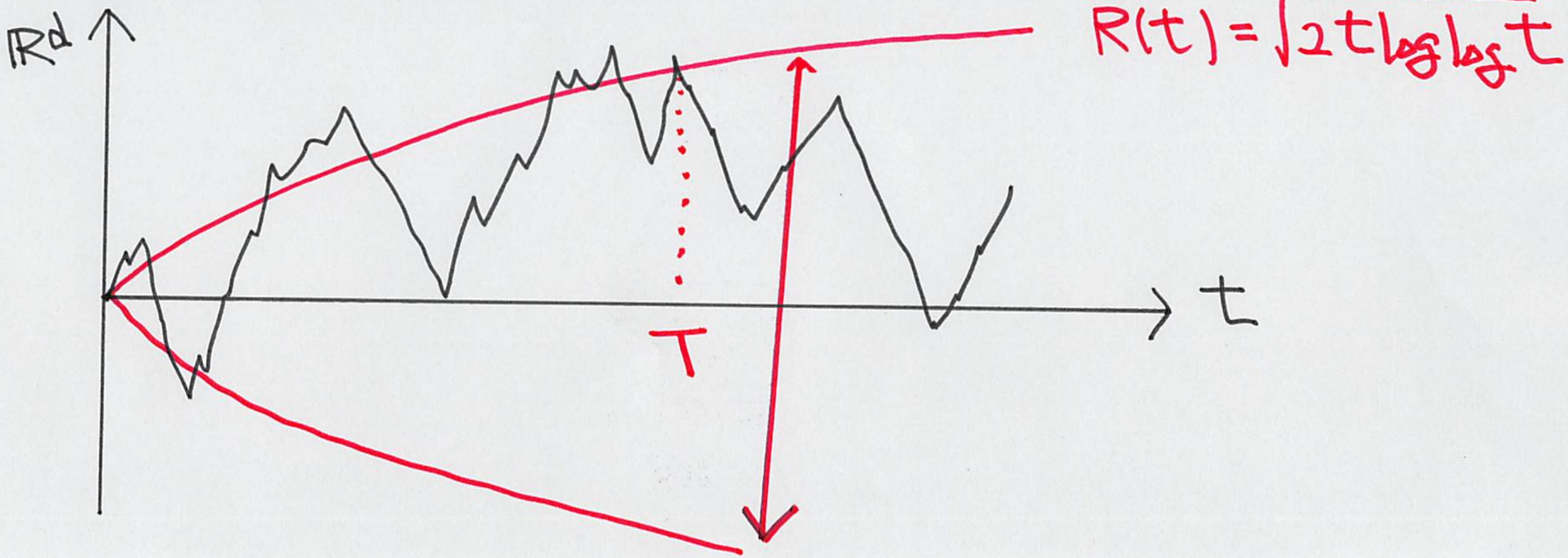
1. Introduction

■ Main interest in this talk:

Range of branching Brownian motions on $\mathbb{R}^d$

▷ $\mathbb{M} = \left(\{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}\right)$: **Brownian motion on** $\mathbb{R}^d$

$$\Rightarrow \quad \limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1 \quad P_x\text{-a.s. (Khintchine's LIL)}$$



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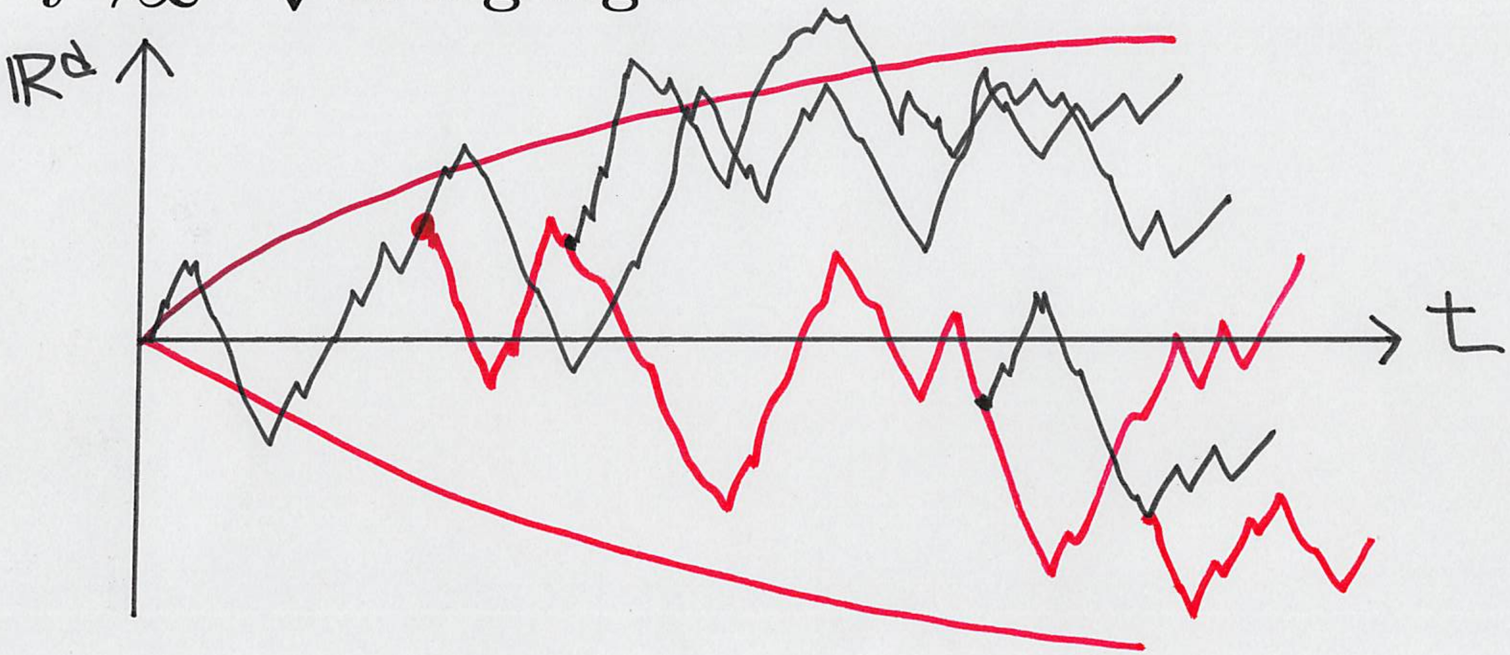

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Q. To determine the **range** of BBMs

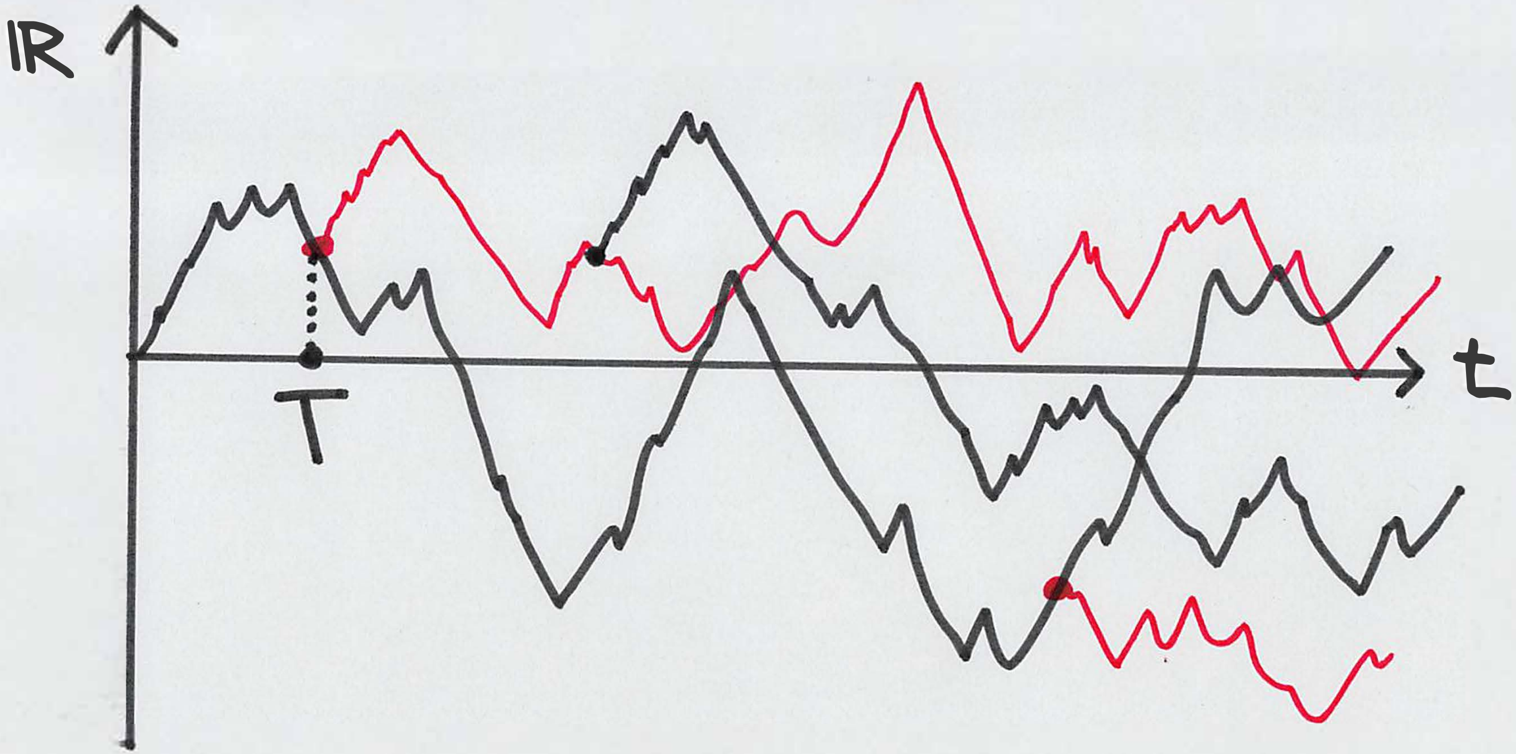
A. Expressed by the eigenvalue of a Schrödinger type operator

Motivation: to reveal the effect of branching to the **range**

■ Homogeneous case for $d = 1$:

* $*$ **Splitting time distribution** $\sim \mathrm{Exp}(c)$ $(c > 0)$
* $*$ **Offspring distribution** $\sim$ **binary branching**

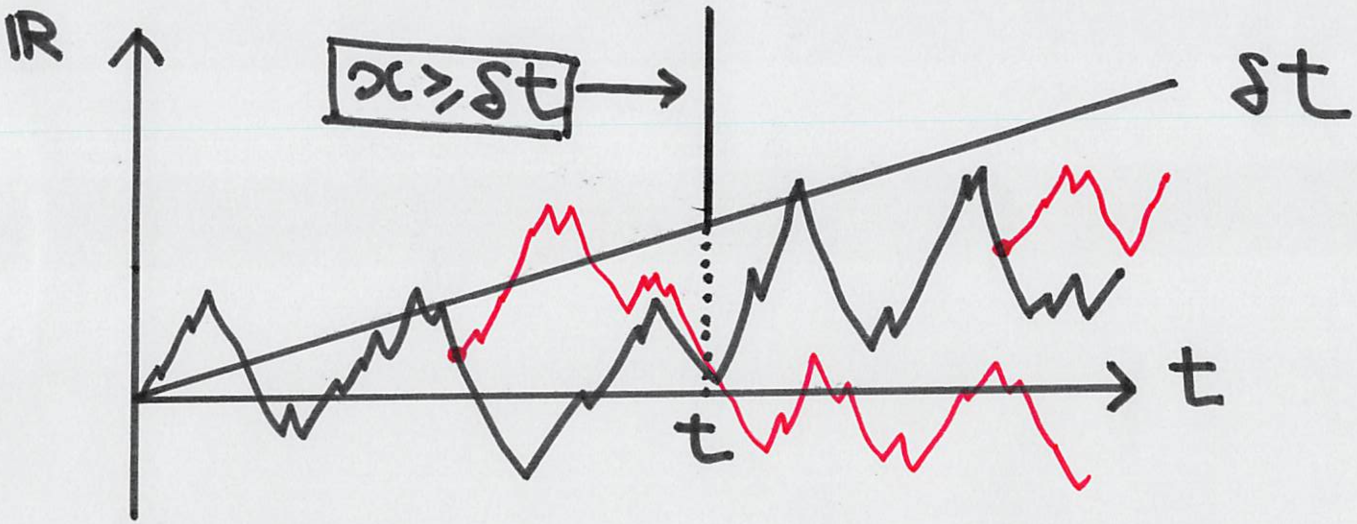
▷ $R_t :=$ **the position of the right most particle at time** t



Homogeneous case for $d = 1$:

Bramson ('78), Roberts ('13) [Mallein ('15) for $d \geq 2$]:

$$R_t \sim \sqrt{2ct} \quad (t \to \infty)$$



▷ $Z_t^{\delta t} :=$ ♯ **of particles on** $\{x \in \mathbb{R} \mid x \geq \delta t\}$

$$\mathbb{E}\left[Z_t^{\delta t}\right] = e^{ct} P(B_t \geq \delta t) \sim \frac{1}{\sqrt{2\pi t}\delta} e^{\left(c - \frac{\delta^2}{2}\right)t} \quad (t \to \infty)$$

Non-homogeneous case for $d = 1$:

* Splitting time distribution

$$P_x(t < T \mid B_s, s \geq 0) = \exp\left(-\int_0^t V(B_s)\, \mathrm{d}s\right)$$

* Offspring distribution $\sim \{p_n(x)\}_{n=2}^{\infty}$

▷ $Q(x) := \sum_{n=2}^{\infty} n p_n(x), \qquad W(x) := (Q(x) - 1)V(x)$

▷ $\lambda := \inf \sigma\left(-\frac{1}{2}\Delta - W\right)$

Erickson ('84): If $Q, V \in C(\mathbb{R})$ and $W(x) \to 0$ ($|x| \to \infty$),

$$\boxed{R_t \sim \sqrt{-\lambda/2}\, t \quad (t \to \infty)}$$

Bocharov-Harris ('14): $V = \beta\delta_0$, $p_2 \equiv 1$ $(\Rightarrow \lambda = -\beta^2/2)$

- $\delta > \beta/2 \ \left(= \sqrt{-\lambda/2}\right) \Rightarrow \lim_{t\to\infty} Z_t^{\delta t} = 0$

- $\delta < \beta/2 \Rightarrow \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\delta t} = \frac{\beta^2}{2} - \beta\delta \ \left(= -\lambda - \sqrt{-2\lambda}\delta\right)$

- $R_t \sim \sqrt{-\lambda/2}t \ (= \beta t/2) \quad (t \to \infty)$

$*$ **Joint distribution of the BM and local time**

Purpose: **to extend the results above to**

(i) more general branching rate

(ii) multidimensional case (without symmetry)

2. Results

■ Model of branching Brownian motions:

* **Splitting time distribution:**

$$P_x(t < T \mid B_s, s \geq 0) = \exp\left(-A_t^{\mu}\right)$$

* **Offspring distribution** $\sim \{p_n(x)\}_{n=2}^{\infty}$

○ A_t^{μ}: **positive continuous additive functional**

* $\mu(\mathrm{d}x) = V(x)\,\mathrm{d}x \Rightarrow A_t^{\mu} = \int_0^t V(B_s)\,\mathrm{d}s$

* $\mu = \delta_0 \Rightarrow A_t^{\delta_0} = 2L(t, 0)$

Kato class measures:

A positive Radon measure μ on $\mathbb{R}^d$ belongs to $\mathcal{K}$ $\overset{\text{def}}{\Longleftrightarrow}$

$$[d=1] \quad \sup_{x\in\mathbb{R}} \int_{|x-y|\le 1} \mu(\mathrm{d}y) < \infty$$

$$[d=2] \quad \lim_{R\to 0} \sup_{x\in\mathbb{R}^2} \int_{|x-y|<R} \log\left(\frac{1}{|x-y|}\right) \mu(\mathrm{d}y) = 0$$

$$[d\ge 3] \quad \lim_{R\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<R} \frac{\mu(\mathrm{d}y)}{|x-y|^{d-2}} = 0$$

○ $p_t^\mu f(x) := E_x\left[e^{A_t^\mu} f(B_t)\right]$: **Feynman-Kac semigroup**

$\Rightarrow \|p_t^\mu\|_{p,q} < \infty \ (1 \le p \le q \le \infty)$ **and strong Feller prop.**

[Albeverio-Blanchard-Ma ('91)]

Model of branching Brownian motions:

∗ **Splitting time distribution:**

$$P_x(t < T \mid B_s, s \geq 0) = \exp\left(-A_t^{\mu}\right) \qquad (\mu \in \mathcal{K})$$

∗ **Offspring distribution** $\sim \{p_n(x)\}_{n=2}^{\infty}$

▷ $Q(x) := \sum_{n=2}^{\infty} n p_n(x)$: **expected offspring number**

at branching cite $x \in \mathbb{R}^d$

▷ $\nu(\mathrm{d}x) := (Q(x) - 1)\,\mu(\mathrm{d}x)$: **intensity of branching**

▷ $Z_t :=$ ♯ of particles at time t

▷ $\mathrm{B}_t = \left(\mathrm{B}_t^1, \ldots, \mathrm{B}_t^{Z_t}\right)$: positions of (Brownian) particles

▷ $Z_t(f) := \sum_{k=1}^{Z_t} f(\mathrm{B}_t^k) \quad (f \in \mathcal{B}_b(\mathbb{R}^d)),\ \ Z_t(A) := Z_t(1_A)$

$$\Rightarrow \mathbb{E}_x[Z_t(f)] = E_x\left[e^{A_t^{\nu}} f(B_t)\right] \ \ [\nu = (Q-1)\mu]$$

▷ $\lambda := \inf \sigma\left(-\frac{1}{2}\Delta - \nu\right)$

$*$ $\boxed{\lambda < 0} \Longrightarrow$ ∃ground state h [Takeda ('03)]

$$\Rightarrow \mathbb{E}_x[Z_t(h)] = E_x\left[e^{A_t^{\nu}} h(B_t)\right] = e^{-\lambda t} h(x)$$

$$\mathbb{E}_x[Z_t(h)] = E_x\left[e^{A_t^{\nu}} h(B_t)\right] = e^{-\lambda t} h(x)$$

▷ $M_t := e^{\lambda t} Z_t(h)$: **non-negative** $\mathbb{P}_x$**-martingale**

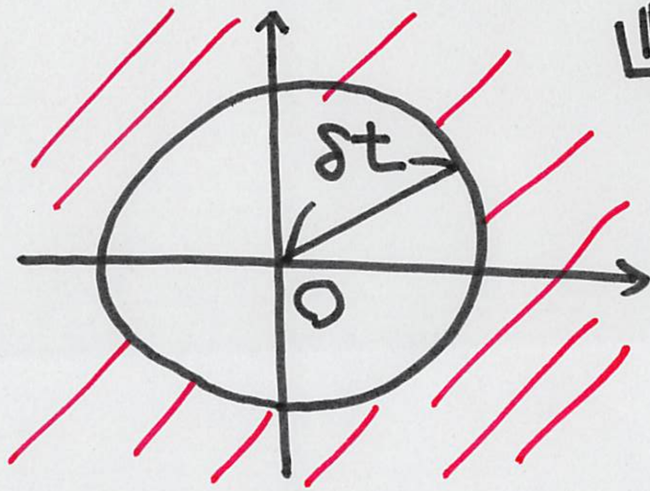
▷ $M_\infty := \lim_{t\to\infty} M_t$

Limit theorem: $\forall A \subset \mathbb{R}^d$: **rel. cpt. open with** $|\partial A| = 0$,

$$Z_t(A) \sim e^{-\lambda t} \int_A h(y)\,\mathrm{d}y\, M_\infty \quad (t \to \infty)$$

[S. Watanabe ('67), Asmussen-Hering ('76),
Z.-Q. Chen-S. ('07), Engländer-Harris-Kyprianou ('10),
Z.-Q. Chen-Y.-X. Ren-T. Yang ('17)]

Result: Population growth rate



▷ $Z_t^{\delta t} := Z_t(\{|x| \geq \delta t\})$

▷ $\nu(\mathrm{d}x) = (Q(x) - 1)\mu(\mathrm{d}x)$

Theorem. **If** $\boxed{\lambda < 0}$ **and** $\underline{\mathrm{supp}[\nu]}$ **is compact, then**

(i) $\delta > \sqrt{-\lambda/2} \Longrightarrow \lim_{t\to\infty} Z_t^{\delta t} = 0,\ \mathbb{P}_x$-**a.s.**

(ii) $0 < \delta < \sqrt{-\lambda/2}$ **and** $\mathbb{P}_x(M_\infty > 0) > 0 \Longrightarrow$

$$\boxed{\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\delta t} = -\lambda - \sqrt{-2\lambda}\delta, \quad \mathbb{P}_x(\cdot \mid M_\infty > 0)\text{-a.s.}}$$

$* \ -\lambda - \sqrt{-2\lambda}\delta > 0 \iff \underline{\delta < \sqrt{-\lambda/2}}$

Result: Spread rate

▷ $L_t := \max_{1 \le k \le Z_t} |\mathrm{B}_t^k|$

Corollary. **Under the same setting as in Theorem,**

$$\lim_{t \to \infty} \frac{L_t}{t} = \sqrt{-\frac{\lambda}{2}}, \quad \mathbb{P}_x(\cdot \mid M_\infty > 0)\text{-a.s.}$$

Remark.

- $$\sup_{x\in\mathbb{R}^d}\sum_{n=1}^{\infty}(n\log n)p_n(x)<\infty \Longrightarrow \mathbb{P}_x(M_\infty>0)>0$$

  [Z.-Q. Chen-Y.-X. Ren-T. Yang ('17)]

- **If we further assume that** $\boxed{d=1,2}$ **and** $\mu(\mathbb{R}^d)<\infty$

  $\Longrightarrow$ $\lambda<0$ **[Takeda ('03)] and** $\mathbb{P}_x(M_\infty>0)=1$ **[S ('08)]**

  $\Longrightarrow$ **Theorem and Corollary hold** $\underline{\mathbb{P}_x\text{-a.s.}}$

Examples (binary branching: $p_2(x) \equiv 1$):

(i) $d = 1,\ \mu = \sum_{n=-\infty}^{\infty} e^{-n^2} \delta_n$

(ii) $d \geq 2,\ \mu = c\delta_{\partial B_R}\ (c > 0)$

$$\boxed{\lambda < 0 \iff cR > (d-2)/2}$$ **[FOT ('11), Takeda ('02)]**

(iii) $\mu(\mathrm{d}x) = cV(x)\,\mathrm{d}x, \quad 0 < V(x) \leq e^{-|x|^p}\ (p > 1)$

- $d = 1, 2 \Rightarrow \lambda < 0$
- $d \geq 3 \Rightarrow \exists c_* > 0$ s.t. $\boxed{\lambda < 0 \iff c > c_*}$

$0 \leq V(x) \leq \frac{1}{|x|^p} \mathbf{1}_{|x| \leq R}$ $\quad \underline{d = 1, p < 1} \quad \underline{d \geq 2, p < 2}$

4. Sketch of the proof

Theorem. If $\boxed{\lambda < 0}$ and $\underline{\mathrm{supp}[\nu] \text{ is compact}}$, then

for $Z_t^{\delta t} := Z_t(\{|x| \geq \delta t\})$,

(i) $\delta > \sqrt{-\lambda/2} \Longrightarrow \lim_{t\to\infty} Z_t^{\delta t} = 0$, $\mathbb{P}_x$-a.s.

(ii) $0 < \delta < \sqrt{-\lambda/2}$ and $\mathbb{P}_x(M_\infty > 0) > 0 \Longrightarrow$

$$\boxed{\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\delta t} = -\lambda - \sqrt{-2\lambda}\delta, \quad \mathbb{P}_x(\cdot \mid M_\infty > 0)\text{-a.s.}}$$

- More general branching rate/mechanism $\Leftarrow$ Upper bound
- Multidimensional case (without symmetry) $\Leftarrow$ Lower bound

Upper bound.

$\circ\ \mathbb{E}_x\left[Z_t^{\delta t}\right] = E_x\left[e^{A_t^\nu} : |B_t| \geq \delta t\right]$

$\triangleright\ \Lambda_\delta := \begin{cases} \lambda + \sqrt{-2\lambda}\,\delta & \text{if } \delta \leq \sqrt{-2\lambda} \\ \delta^2/2 & \text{if } \delta > \sqrt{-2\lambda} \end{cases}$

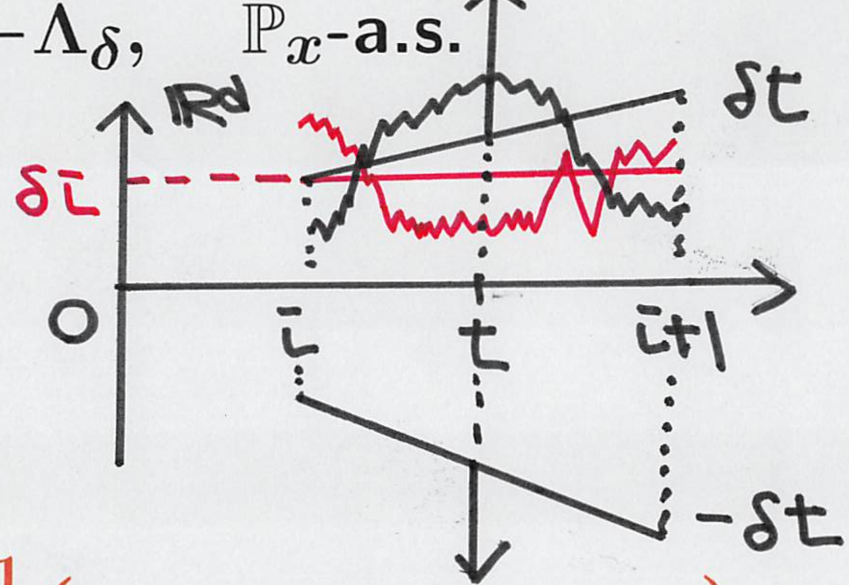
Key Fact: If $\lambda < 0$ and $\mathrm{supp}[\nu]$ is compact, then

$$E_x\left[e^{A_t^\nu} : |B_t| \geq \delta t\right] \approx \left(e^{-\lambda t}\int_{|y|\geq \delta t} h(y)\,\mathrm{d}y\right) \wedge P_x(|B_t| \geq \delta t) \approx e^{-\Lambda_\delta t}$$

- $\delta = 0$: L^p-indep. spectral bounds [Takeda ('98, '03, '08)]

Lemma. **Under the full conditions as in Theorem,**

$$\limsup_{t\to\infty} \frac{1}{t}\log Z_t^{\delta t} \le -\Lambda_\delta, \quad \mathbb{P}_x\text{-}\mathbf{a.s.}$$



* **Borel-Cantelli lemma**

- $i \le t < i+1$

$$Z_t^{\delta t} = \sum_{k=1}^{Z_t} 1_{\{|\mathrm{B}_t^k| \ge \delta t\}} \le \sum_{k=1}^{Z_{i+1}} 1_{\left\{\sup\limits_{i\le s<i+1} |\mathrm{B}_s^k| \ge \delta i\right\}}$$

$$\boxed{\le}\ e^{(-\Lambda_\delta+\varepsilon)i} \le ce^{(-\Lambda_\delta+\varepsilon)t}$$

$$\Longrightarrow \quad \limsup_{t\to\infty} \frac{1}{t}\log Z_t^{\delta t} \le -\Lambda_\delta, \quad \mathbb{P}_x\text{-}\mathbf{a.s.}$$

Remark.

(1) $\triangleright\ L_t^{\mathrm{r}} := \max_{1\le k\le Z_t} \langle \mathrm{B}_t^k, \mathrm{r}\rangle \quad (\mathrm{r}\in\mathbb{R}^d$: unit vector)

Under the same setting as in Theorem,

$$\lim_{t\to\infty}\frac{L_t^{\mathrm{r}}}{t} = \sqrt{-\frac{\lambda}{2}}, \quad \mathbb{P}_x(\cdot \mid M_\infty > 0)\text{-a.s.}$$

(2) $d \ge 3$ and $\boxed{\lambda = 0}$: If $\mathrm{supp}[\mu]$ is compact and

$$\sup_{x\in\mathbb{R}^d}\sum_{n=2}^{\infty} n p_n(x) < \infty, \text{ then}$$

$$\limsup_{t\to\infty}\frac{L_t}{\sqrt{2t\log\log t}} = 1, \quad \mathbb{P}_x\text{-a.s.}$$

$\ast$ local extinction [S ('06, '08)]