Bottom crossing probability for symmetric jump processes

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1. Introduction

■ **Main interest in this talk:**

To understand in detail transience/non-point recurrence

for symmetric jump processes

$\rightsquigarrow$ Quantification (Lower rate functions)

▷ Bottom crossing probability: related tail probability

Purpose:

to determine the decay rate of the bottom crossing probability

▷ $\mathbb{M} = \left(\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}\right)$: **symm. Lévy proc. on** $\mathbb{R}^d$

Definition.

(i) $\mathbb{M}$ **is transient** $\iff$

$$P\left(\lim_{t \to \infty} |X_t| = \infty\right) = 1$$

(ii) $\mathbb{M}$ **is non-point recurrent**

$$P\left(|X_t| > 0 \text{ for all } t > 0 \text{ and } \liminf_{t \to \infty} |X_t| = 0\right) = 1$$

▷ $\mathbb{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}\right)$: **symm. Lévy proc. on** $\mathbb{R}^d$

$$P\left(\lim_{t\to\infty} |X_t| = \infty\right) = 1$$

$\Rightarrow$ **how fast the particle goes to infinity**

$$P\left(|X_t| > 0 \text{ for all } t > 0 \text{ and } \liminf_{t\to\infty} |X_t| = 0\right) = 1$$

$\Rightarrow$ **how arbitrary close the particle comes to the origin**

Quantitative expression

▷ $r(t)$: positive function on $(0, \infty)$

Definition.

$r(t)$ is a **lower rate function** for $\mathbb{M}$ $\iff$

$$P\left(\exists T > 0 \text{ s.t. } |X_t| > r(t) \text{ for all } t \geq T\right) = 1$$

- $r(t)$: **bottom** of $|X_t|$ for all sufficiently large time

Quantitative expression of transience/non-point recurrence

▷ $T := \sup\{t > 0 : |X_t| \le r(t)\}$

$$P(T > t) = P\left(\exists s > t \text{ s.t. } |X_s| \le r(s)\right) \to 0 \ (t \to \infty)$$

Purpose: to find the decay rate of $P(T > t)$ as $t \to \infty$

under more general setting

Motivation:

to reveal the relation between $r(t)$ and the tail of T

$r(t)$: large $\Rightarrow$ tail of T: large

∘ Use Borel-Cantelli's lemma $\Rightarrow$ difficult to find the tail of T

▷ $\mathbb{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d}\right)$: **symm. α-stable proc. on $\mathbb{R}^d$**

▷ $\alpha \in (0, 2]$

▷ $r(t) = t^{1/\alpha} g(t)$ **($g(t) \searrow 0$ as $t \to \infty$ and some cond.)**

▷ $I(t) := \displaystyle\int_t^\infty g(s)^{d-\alpha} \frac{\mathrm{d}s}{s}$

Theorem. **Suppose $d > \alpha$ (transience).**

(i) [Dvoretzky-Erdős ('51), Takeuchi ('64)]

If $\exists t_0 > 0$ s.t. $I(t_0) < \infty$ (or $= \infty$)

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |X_t| \geq r(t) \text{ for all } t \geq T\right) = 1 \text{ (or } 0)$$

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(ii) [Wichura ('79)]

If $\exists t_0 > 0$ s.t. $I(t_0) < \infty \Rightarrow \exists C_{d,\alpha} > 0$ s.t.

$$P\left(\exists s > t \text{ s.t. } |X_s| \leq r(s)\right) \sim C_{d,\alpha} I(t) \quad (t \to \infty)$$

Example.

$$\triangleright\ r(t) = \frac{t^{\frac{1}{\alpha}}}{(\log t)^{\frac{1+\varepsilon}{d-\alpha}}} \qquad \left(\Longrightarrow g(t) = \frac{1}{(\log t)^{\frac{1+\varepsilon}{d-\alpha}}} \right)$$

$$\Longrightarrow \qquad I(t) = \int_t^\infty g(s)^{d-\alpha} \frac{\mathrm{d}s}{s} = \int_t^\infty \frac{\mathrm{d}s}{s(\log s)^{1+\varepsilon}}$$

Hence

$$\boxed{I(t) < \infty \iff \varepsilon > 0}$$

Moreover, if $\varepsilon > 0$, then

$$P\left(\exists s > t \text{ s.t. } |X_s| \le r(s)\right) \sim \frac{C_{d,\alpha}}{\varepsilon} \frac{1}{(\log t)^{\varepsilon}} \quad (t \to \infty)$$

- **Spitzer ('58), Takeuchi-S. Watanabe ('64), Wichura ('79)**

  non-point recurrent case $\boxed{d = \alpha \in \{1, 2\}}$

- **S.-J. Wang ('16+): rate functions via heat kernel**

  $*$ **sharp criterion/weak scaling**

  $*$ **cover symm. α-stable-like proc. with** $\boxed{\alpha > 2}$

■ **Result and consequence in this talk:**

Extension of Wichura ('79) to the setting of S.-J. Wang

$\Rightarrow$ **to reveal how the global/local properties affect the BCP**

2. Result

▷ M: locally compact separable metric space

▷ μ: positive Radon measure on M with full support

▷ $\mathbb{M} = \big(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in M}\big)$: μ-symm. Hunt proc. on M

Suppose that $\mathbb{M}$ generates a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$:

$$\mathcal{E}(u,u) = \iint_{M\times M} (u(x) - u(y))^2 J(x,y)\,\mu(\mathrm{d}x)\mu(\mathrm{d}y)$$

($J(x,y)$: positive, symmetric function)

▷ $B(x,r) := \{y \in M : d(y,x) < r\}$

▷ $V(x,r) := \mu(B(x,r))$: **volume of the ball**

Assumption.

(i) $\exists c_1, c_2, d_1, d_2 > 0$ **s.t.** $\forall x \in M$,

$$c_1 \left(\frac{R}{r}\right)^{d_1} \le \frac{V(x,R)}{V(x,r)} \le c_2 \left(\frac{R}{r}\right)^{d_2} \quad (0 < r < R)$$

(ii) $\exists$**nonneg. symm. kernel** $p(t,x,y)$ **on** $(0,\infty) \times M \times M$ **s.t.**

- $P_x(X_t \in A) = \int_A p(t,x,y)\,\mu(\mathrm{d}y), \quad A \in \mathcal{B}(M)$
- $p(t+s,x,y) = \int_M p(t,x,z)p(s,z,y)\,\mu(\mathrm{d}z)$

(iii) $\exists\phi$: **increasing function on** $[0,\infty)$ **s.t.**

- $\phi(0) = 0$
- $\exists c_3, c_4, d_3, d_4 > 0$ **s.t.**

$$c_3\left(\frac{R}{r}\right)^{d_3} \le \frac{\phi(R)}{\phi(r)} \le c_4\left(\frac{R}{r}\right)^{d_4} \quad (0 < r < R)$$

$$p(t,x,y) \asymp \frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}$$

Example. **(symm. stable-like proc. [Z.-Q. Chen-Kumagai '03])**

$V(x,r) = r^{\alpha},\ \phi(r) = r^{\beta}\ (\alpha,\beta > 0)$

$$\Longrightarrow\ p(t,x,y) \asymp \frac{1}{t^{\beta/\alpha}} \wedge \frac{t}{d(x,y)^{\beta+\alpha}}$$

Remark. **[Z.-Q. Chen-Kumagai-J. Wang ('16)]**

Under Assumption,

- $\mathbb{M}$ **is conservative;**
- $J(x,y) \asymp \dfrac{1}{V(x, d(x,y))\phi(d(x,y))}$

Theorem.

▷ $r(t) := \phi^{-1}(t)g(t)$ **(**$g(t) \searrow 0$ **as** $t \to \infty$**) and some cond.**

$$I(t) := \int_t^\infty \frac{V(x, r(s))}{\phi(r(s))} \frac{\mathrm{d}s}{V(x, \phi^{-1}(s))} < \infty$$

$$\Longrightarrow \quad P_x(\exists s > t \text{ s.t. } d(x, X_s) \le r(s)) \asymp I(t) \quad (t \to \infty)$$

Example.

▷ $V(x,r) \asymp r^{\alpha_1} 1_{\{r<1\}} + r^{\alpha_2} 1_{\{r\geq 1\}}$ $(\alpha_1, \alpha_2 > 0)$

▷ $\phi(r) \asymp r^{\beta_1} 1_{\{r<1\}} + r^{\beta_2} 1_{\{r\geq 1\}}$ $(\beta_1, \beta_2 > 0)$

- **Symmetric stable-like processes of variable order**
- **Subordinate diffusion proc. with sub-Gauss. HK estimates [Bogdan-Stós-Sztonyk ('03), Kumagai ('03)]**

Assume $\alpha_1 > \beta_1$, $\alpha_2 > \beta_2$ ($\Rightarrow \mathbb{M}$: transient). Then

$$I(t) \asymp \int_t^\infty \frac{V(x, r(s))}{\phi(r(s))} \frac{\mathrm{d}s}{s^{\frac{\alpha_2}{\beta_2}}} \quad (t \to \infty)$$

$\triangleright\ q_r(t,x) := P_x(\exists s > t \text{ s.t. } d(x,X_s) \le r(s)) \asymp I(t)$

$\circ\ r(t) = \dfrac{t^{1/\beta_2}}{(\log t)^{\frac{1+\varepsilon}{\alpha_2-\beta_2}}} \quad (\varepsilon > 0) \Longrightarrow$

$$q_r(t,x) \asymp \frac{1}{\varepsilon(\log t)^{\varepsilon}} \quad (t \to \infty)$$

$\circ\ r(t) = t^p \ \left(p < \dfrac{1}{\beta_2}\right) \Longrightarrow$

$$q_r(t,x) \asymp \begin{cases} \dfrac{1}{t^{\left(\frac{1}{\beta_2}-p\right)(\alpha_2-\beta_2)}} & \left(0 \le p < \dfrac{1}{\beta_2}\right) \\ \dfrac{1}{t^{\frac{1}{\beta_2}(\alpha_2-\beta_2)-p(\alpha_1-\beta_1)}} & (p < 0) \end{cases}$$

3. Sketch of the proof

Theorem.

▷ $r(t) := \phi^{-1}(t)g(t)$ $\quad$ ($g(t) \searrow 0$ as $t \to \infty$) and some cond.

$$I(t) := \int_t^\infty \frac{V(x, r(s))}{\phi(r(s))} \frac{\mathrm{d}s}{V(x, \phi^{-1}(s))} < \infty$$

$$\Longrightarrow \; P_x(\exists s > t \text{ s.t. } d(x, X_s) \leq r(s)) \asymp I(t) \quad (t \to \infty)$$

Upper bound.

▷ $n_k = tc^k \; (c > 1)$

▷ $A_k = \big\{\exists s \in (n_k, n_{k+1}] \text{ s.t. } d(X_0, X_s) \leq r(s)\big\}$

Then

$$
\begin{aligned}
&P_x(\exists s > t \text{ s.t. } d(x, X_s) \leq r(s)) \\
&= P_x\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} P_x(A_k)
\end{aligned}
$$

$$
P_x(A_k) = P_x\big(\exists s \in (n_k, n_{k+1}] \text{ s.t. } d(x, X_s) \leq r(s)\big)
$$

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Lemma 1. **[Khoshnevisan ('97), S.-J. Wang ('16+)]**

For any $b > a > 0$, $c > 0$, $r > 0$,

$$P_x\left(\exists s \in (a, b] \text{ s.t. } d(x, X_s) \le r\right) \le \frac{\int_a^{b+c} \boxed{P_x(d(x, X_s) \le 2r)}\, \mathrm{d}s}{\int_0^c \inf_{y \in M : d(y,x) \le r} \boxed{P_y(d(y, X_s) \le r)}\, \mathrm{d}s}$$

$$P_x(d(x, X_t) \le r) \asymp \underbrace{1}_{\boxed{\phi(r) > t}} \wedge \underbrace{\frac{V(x, r)}{V(x, \phi^{-1}(t))}}_{\boxed{\phi(r) \le t}},$$

If $\phi(r) \leq a \wedge c$, then

$$P_x\left(\exists s \in (a,b] \text{ s.t. } d(x,X_s) \leq r\right) \lesssim \frac{V(x,r)}{\phi(r)} \int_a^{b+c} \frac{\mathrm{d}s}{V(x,\phi^{-1}(s))}$$

Lemma 2.

$\forall c \in (1,2), \exists T_c > 0$ s.t. for all $t \geq T_c$,

$$P_x(A_k) = P_x\left(\exists s \in (n_k, n_{k+1}] \text{ s.t. } d(x,X_s) \leq r(s)\right) \leq K_{c,g,t} \int_{n_k}^{n_{k+1}} \frac{V(x,r(u))}{\phi(r(u))} \frac{\mathrm{d}u}{V(x,\phi^{-1}(u))}$$

We get the upper bound by this lemma.