対称マルコフ過程の大域的性質とディリクレ形式

塩沢 裕一
（岡山大学自然・環境理工）

日本数学会秋季総合分科会
京都産業大学
2015年9月13日

1. Introduction

Main interest:

Global path properties of symmetric Markov processes

conservativeness/transience/recurrence

- Criteria
- Quantitative characterizations

(⇝ upper/lower rate functions)

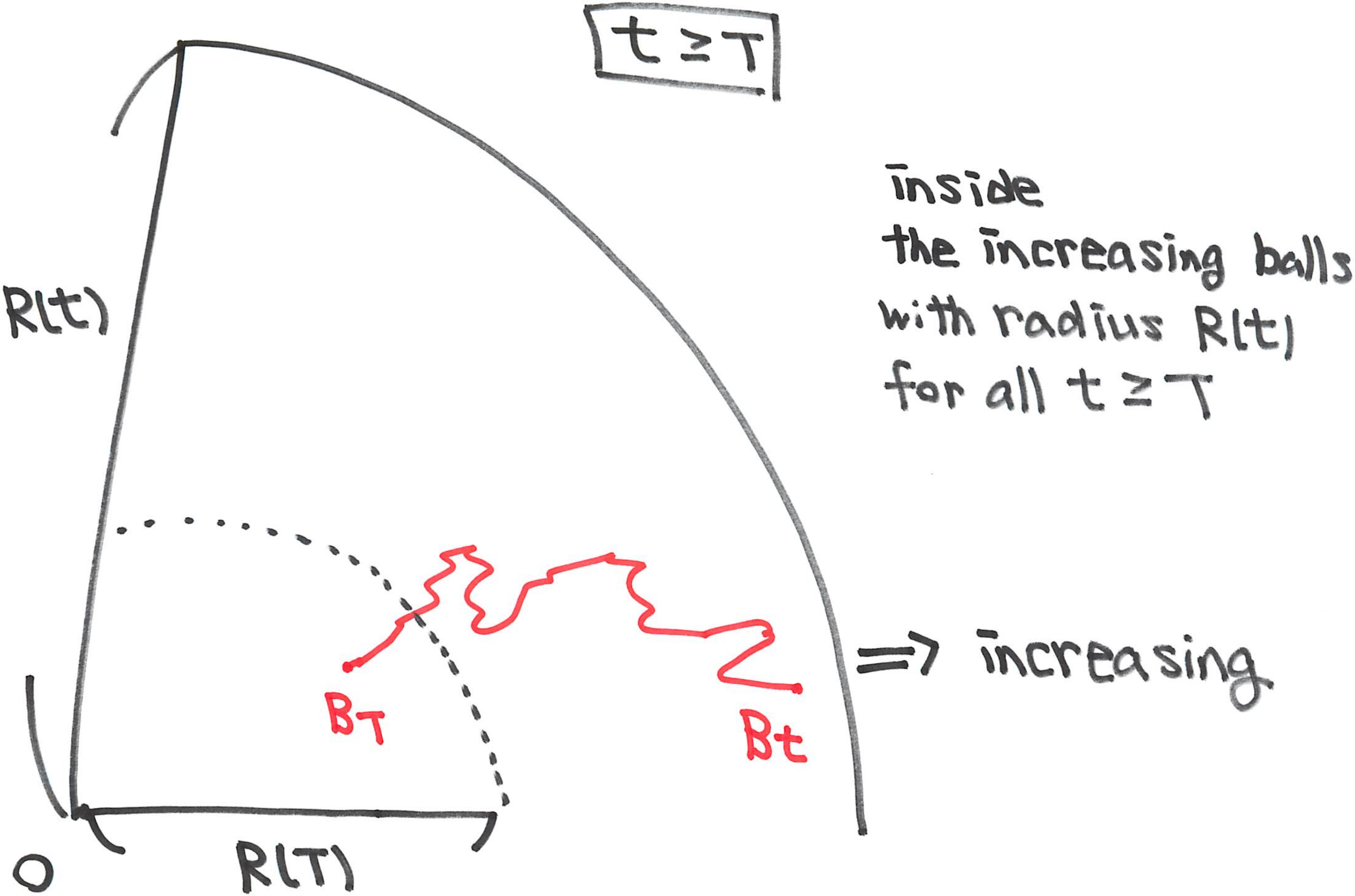
▷ $(\{B_t\}_{t\geq 0}, P)$: **Brownian motion on** $\mathbb{R}^d$, $B_0 = 0$ **a.s.**

Kolmogorov's test **(e.g., see Itô-McKean)**

▷ $R(t) = \sqrt{t}g(t)$ **(**$g(t) \nearrow \infty$ **as** $t \to \infty$**)**

$$\text{(U)} \quad \int_{\cdot}^{\infty} \left(\frac{R(t)^2}{t}\right)^{d/2} \exp\left(-\frac{R(t)^2}{2t}\right) \frac{dt}{t} < \infty \ (\text{or} = \infty)$$

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |B_t| \leq R(t) \text{ for all } t \geq T\right) = 1 \ (\text{or } 0)$$


t ≥ T
R(t)
inside
the increasing balls
with radius R(t)
for all t ≥ T
⟹ increasing
B_T
B_t
O
R(T)

▷ $(\{B_t\}_{t\geq 0}, P)$: **Brownian motion on** $\mathbb{R}^d$, $B_0 = 0$ **a.s.**

Kolmogorov's test (e.g., see Itô-McKean)

▷ $R(t) = \sqrt{t}g(t)$ **(**$g(t) \nearrow \infty$ **as** $t \to \infty$**)**

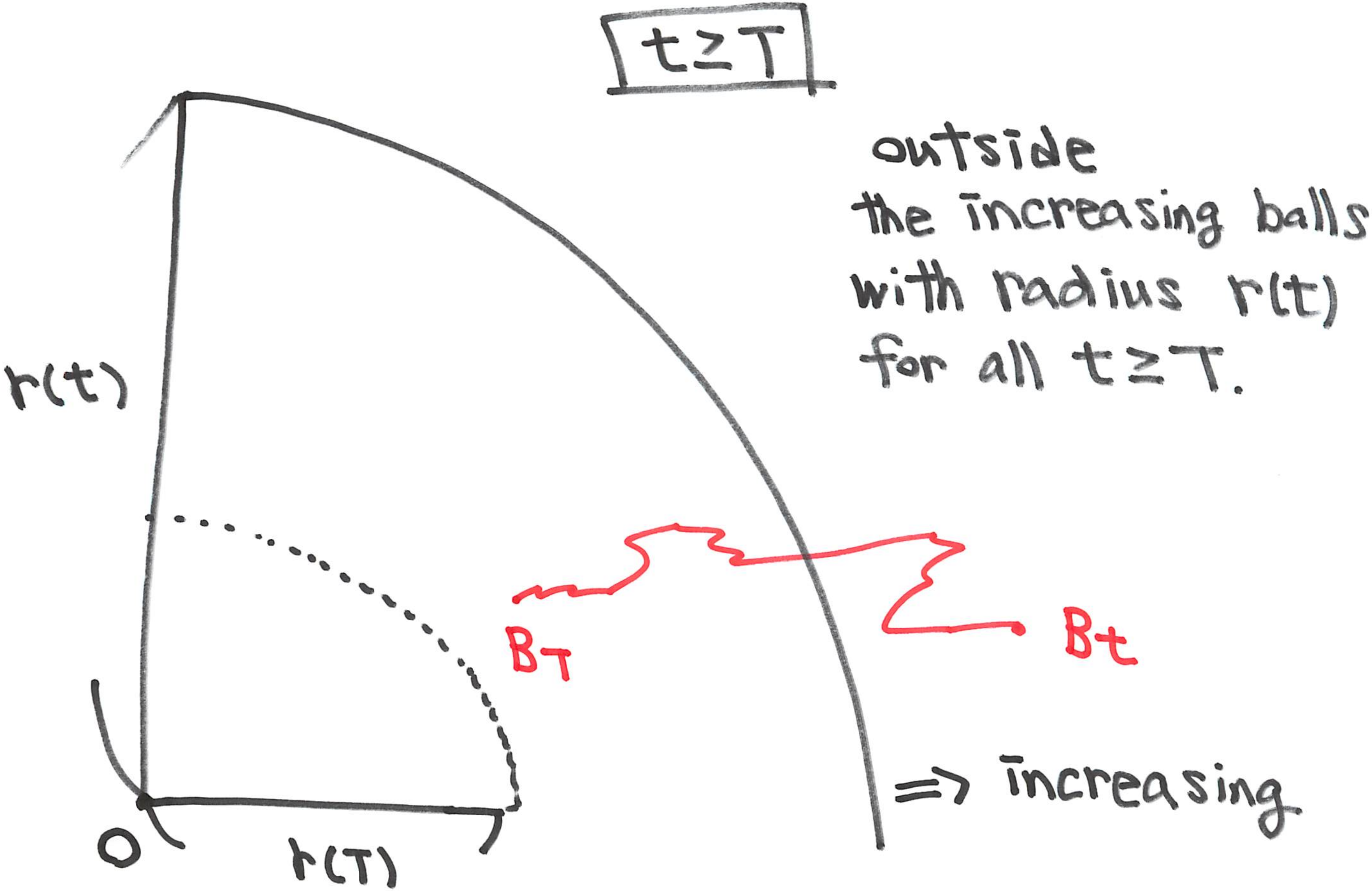
$$\text{(U)} \quad \int_{\cdot}^{\infty} g(t)^d \exp\left(-\frac{g(t)^2}{2}\right) \frac{\mathrm{d}t}{t} < \infty \ (\text{or} = \infty)$$

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |B_t| \leq R(t) \text{ for all } t \geq T\right) = 1 \ (\text{or } 0)$$

• $R(t)$: **upper rate function for the one-probability case**

Example. $R(t) = \sqrt{(2+\varepsilon)t \log\log t}$

$$\boxed{\text{(U)} \iff \varepsilon > 0}$$



t≥T
outside the increasing balls with radius r(t) for all t≥T.
r(t)
B_T
B_t
⟹ increasing
O
r(T)

Dvoretzky-Erdös' test ('51) $[d \geq 3]$

▷ $r(t) = \sqrt{t}h(t)$ ($h(t) \searrow 0$ as $t \to \infty$)

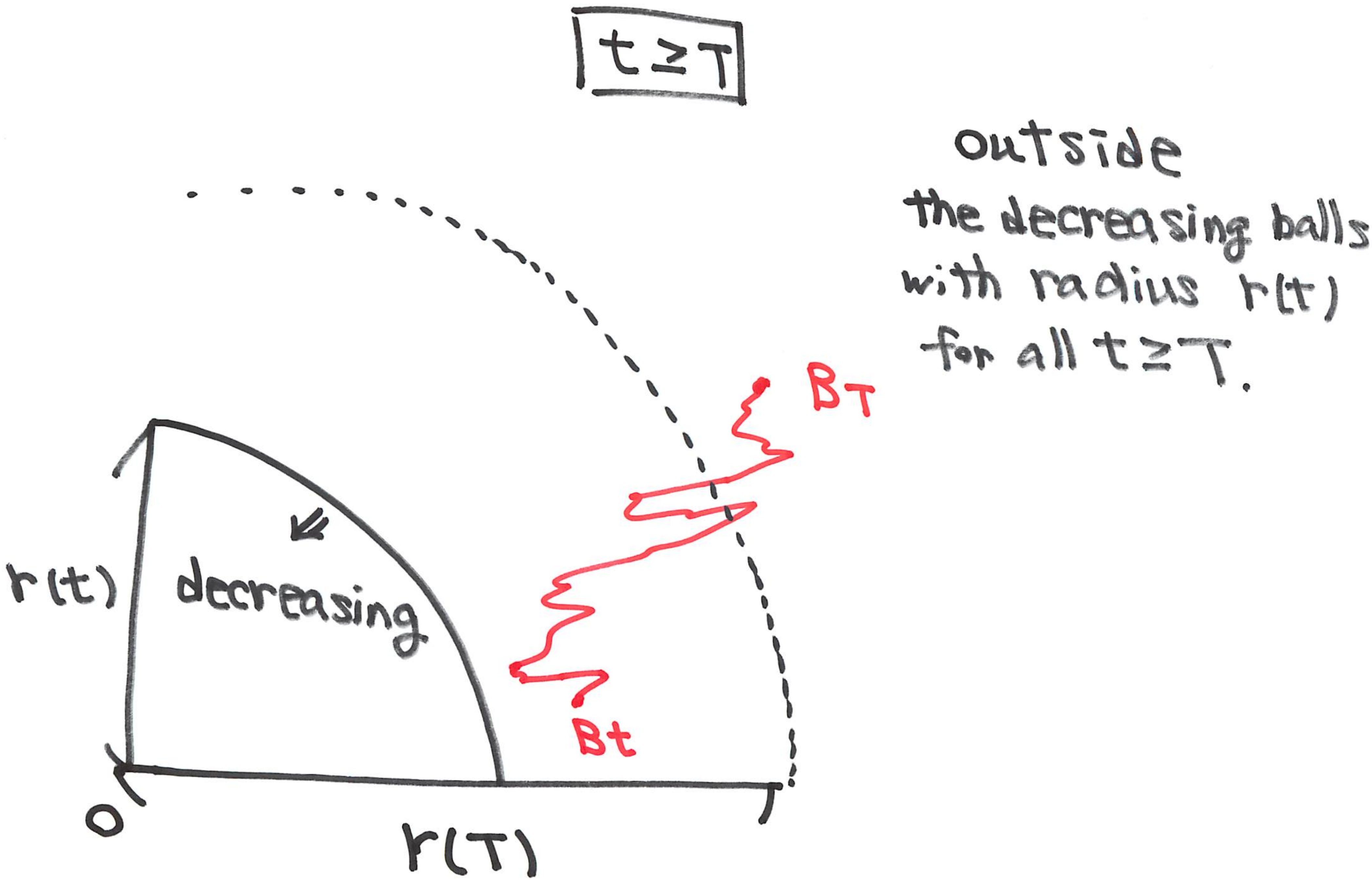
$$(\mathrm{L}_1) \quad \int_{\cdot}^{\infty} h(t)^{d-2} \frac{\mathrm{d}t}{t} < \infty \ (\mathrm{or} = \infty)$$

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |B_t| \geq r(t) \text{ for all } t \geq T\right) = 1 \ (\mathrm{or}\ 0)$$

• $r(t)$: lower rate function for the one-probability case

Example. $r(t) = \dfrac{\sqrt{t}}{(\log t)^{\frac{1+\varepsilon}{d-2}}}$

$$\boxed{(\mathrm{L}_1) \iff \varepsilon > 0}$$



t ≥ T
outside the decreasing balls with radius r(t) for all t ≥ T.
B_T
r(t)
decreasing
Bt
0
r(T)

Spitzer's test ('58) $[d = 2]$

$\triangleright$ $r(t) = \sqrt{t}h(t)$ ($h(t) \searrow 0$ as $t \to \infty$)

$$(\mathrm{L}_2) \quad \int_{\cdot}^{\infty} \frac{\mathrm{d}t}{t|\log h(t)|} < \infty \ (\text{or} = \infty)$$

$\Longrightarrow P\,(\exists T > 0$ s.t. $|B_t| \geq r(t)$ for all $t \geq T) = 1$ (or 0)

- $r(t)$: lower rate function for the one-probability case

Example. $r(t) = \dfrac{1}{t^{(\log\log t)^{1+\varepsilon}}}$

$$\boxed{(\mathrm{L}_2) \iff \varepsilon > 0}$$

* **Symmetric α-stable processes on $\mathbb{R}^d$ $(0 < \alpha < 2)$**

Upper: Khintchine ('38) $\left[R(t) = t^{\frac{1}{\alpha}}(\log t)^{\frac{1+\varepsilon}{\alpha}}\right]$

Lower: Takeuchi ('64) $(d > \alpha)$

$$\left[r(t) = \frac{t^{\frac{1}{\alpha}}}{(\log t)^{\frac{1+\varepsilon}{d-\alpha}}}\right]$$

Takeuchi-S. Watanabe ('64) $(d = \alpha = 1)$

$$\left[r(t) = \frac{1}{t^{(\log\log t)^{1+\varepsilon}}}\right]$$

Hendricks ('70), Khoshnevisan ('97)

Dirichlet form: Axiomatization of the Dirichlet integral

$$\mathbb{D}(u,v) = \int_{\mathbb{R}^d} \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x, \quad u,v\in H^1(\mathbb{R}^d)$$

$(\mathbb{D}/2, H^1(\mathbb{R}^d)) \iff$ **Brownian motion on** $\mathbb{R}^d$

- X: **locally compact separable metric space**
- m: **positive Radon measure on** X **with full support**
- $(\mathcal{E},\mathcal{F})$: **regular Dirichlet form on** $L^2(X;m) \Longrightarrow$

$\exists\mathbb{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in X}\right)$: **symmetric Hunt process on** X

[Fukushima ('71), Fukushima-Oshima-Takeda ('11)]

Example.

▷ $\{a_{ij}(x)\}_{i,j=1}^{d}$: **Borel measurable functions on** $\mathbb{R}^d$ **s.t.**

- $a_{ij}(x) = a_{ji}(x) \quad (1 \le i, j \le d)$
- $\displaystyle\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \asymp |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^d$

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)\, \mathrm{d}x, \quad u \in H^1(\mathbb{R}^d)$$

$\rightsquigarrow (\mathcal{E}, \mathcal{F}) = \Big(\mathcal{E}, H^1(\mathbb{R}^d)\Big)$: **regular Dirichlet form on** $L^2(\mathbb{R}^d)$

• **Symmetric stable(-like) processes on $\mathbb{R}^d$:**

▷ $\mathcal{L} = -\frac{1}{2}(-\Delta)^{\alpha/2} \quad (\alpha \in (0,2))$

$$\mathcal{E}(u,u) = c_{d,\alpha} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

$$= (-\mathcal{L}u, u)_{L^2(\mathbb{R}^d)}$$

▷ $c(x,y) \asymp 1$: **Borel measurable**

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

[Z.-Q. Chen-Kumagai ('03)]

Beurling-Deny formula

$$\mathcal{E}(u,u) = \mathcal{E}^{(c)}(u,u)$$

$$+ \iint_{X\times X\setminus \mathrm{diag}} (u(x)-u(y))^2 \, J(\mathrm{d}x\mathrm{d}y) + \int_X u(x)^2 \, k(\mathrm{d}x)$$

- $\mathcal{E}^{(c)}$: symm. form with the strong local property
- $J(\mathrm{d}x\mathrm{d}y)$: jumping measure
- k: killing measure

$\Longrightarrow$ continuity of sample paths/killing inside

- **Symmetric diffusion processes**

$$\mathcal{E}(u,u) = \mathcal{E}^{(c)}(u,u) = \frac{1}{2}\int_X \mathrm{d}\mu^{(c)}_{\langle u\rangle}, \quad u \in \mathcal{F}$$

$$\text{“}\mu^{(c)}_{\langle u\rangle}(\mathrm{d}x) \asymp \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\,\mathrm{d}x\text{”}$$

  - **Volume growth rate (symmetrizing measure m)**

  - **Coefficient growth/decay rate**

Upper: **Takeda ('89), Grigor'yan-Hsu ('08), Hsu-Qin ('10), Ouyang ('13)**

Lower: **Ichihara ('78)**

Both: **Grigor'yan-Kelbert ('98), Grigor'yan ('99), Bendikov-Saloff-Coste ('05)**

Assumption. $\exists \rho : X \to [0, \infty)$ **such that**

(i) $\rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X)$ **and** $\rho(x) \to \infty$ **as** $x \to \Delta$

(ii) $\exists \Gamma(\rho) = \dfrac{\mathrm{d}\mu^{(c)}_{\langle \rho \rangle}}{\mathrm{d}m}$ $\left(\text{“}\Gamma(\rho) = \sum_{i,j=1}^{d} a_{ij} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j}\text{”} \right)$

(iii) $B_\rho(r) = \{x \in X \mid \rho(x) \le r\}$**: compact** $(\forall r > 0)$

$\triangleright$ $\lambda_\rho(r) = \sup_{x \in B_\rho(r)} \Gamma(\rho)(x)$

$\triangleright$ $\psi(R) = \displaystyle\int_6^R \frac{r}{\lambda_\rho(r)(\log m(B_\rho(r)) + \log \log r)} \mathrm{d}r$

cf. Grigor'yan ('99): $\psi(R) = \dfrac{R^2}{\log m(B(R))}$ $(= t)$

If $\lim_{R\to\infty} \psi(R) = \infty$, **then** $\exists c > 0$ **s.t. for** m**-a.e.** $x \in X$**,**

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le \psi^{-1}(ct) \text{ for all } t \ge T\right) = 1$$

$$(ct =)\ \psi(R) = \int_6^R \frac{r}{\lambda_\rho(r)(\log m(B_\rho(r)) + \log\log r)}\,\mathrm{d}r$$

Remark. **(1)** **Hsu-Qin ('10): add** "$\log\log R$"

(2) Intrinsic metric

Biroli-Mosco ('91), Sturm ('94)

Example.

▷ $\mathcal{E}(u,u) \asymp \int_{\mathbb{R}^d} (1+|x|)^p |\nabla u(x)|^2 \,\mathrm{d}x$

$$\rho(x) = \int_0^{|x|} \frac{\mathrm{d}r}{\sqrt{(1+r)^p}} \asymp \begin{cases} |x|^{1-p/2} & (0 \le p < 2) \\ \log(2+|x|) & (p = 2) \end{cases}$$

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le \psi^{-1}(ct) \text{ for all } t \ge T\right) = 1$$

$$\Longrightarrow P_x\left(\exists T > 0 \text{ s.t. } |X_t| \le R(Ct) \text{ for all } t \ge T\right) = 1$$

- $0 \le p < 2 \Longrightarrow R(t) \asymp (t \log t)^{\frac{1}{2-p}}$
- $p = 2 \Longrightarrow R(t) \asymp e^{c_1 t}$

Main interest.

Rate functions for symmetric Markov processes

- Rate functions under volume/coefficient growth rates

  Upper: Huang ('14), Huang-S. ('14), S ('15+)

  Lower: S ('15+) (transient)

- 0-1 laws for rate functions under heat kernel estimates

  Both: S.-J. Wang ('15) (transient/critical case)

2. Rate functions under volume/coefficient growth rate

▷ (X, d): **locally compact separable metric space**

▷ m: **positive Radon measure on X with full support**

▷ $(\mathcal{E}, \mathcal{F})$: **regular Dirichlet form s.t.**

$$\mathcal{E}(u,u) = \iint_{X \times X \setminus \mathrm{diag}} (u(x) - u(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$

▷ $J(x, \mathrm{d}y)$: **jump kernel s.t.**

$$J(x, \mathrm{d}y) m(\mathrm{d}x) = J(y, \mathrm{d}x) m(\mathrm{d}y)$$

▷ $\mathbb{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in X})$: **associated symm. Hunt proc.**

Upper rate functions

$$\iint_{X\times X\setminus \mathrm{diag}} = \underbrace{\iint_{d(x,y)<1}}_{\textbf{small jump}} + \underbrace{\iint_{d(x,y)\geq 1}}_{\textbf{big jump (perturbation)}}$$

$$\sup_{x\in X}\int_{d(x,y)\geq 1} J(x,\mathrm{d}y) < \infty$$

Conservativeness criterion

Masamune-Uemura ('11)

Grigor'yan-Huang-Masamune ('12)

Masamune-Uemura-J. Wang ('12): $1 \to c > 0$

S.-Uemura ('14), S. ('15): $1 \to F(x,y) > 0$

- $\mathcal{A} = \left\{ \rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X) : \begin{array}{l} \lim_{x \to \Delta} \rho(x) = \infty \text{ and} \\ B_\rho(r) \text{ is compact } (\forall r > 0) \end{array} \right\}$
- $B_\rho(r) = \{x \in X \mid \rho(x) \leq r\}$

$$\boxed{d(x,y) < F(x,y) \Longrightarrow |\rho(x) - \rho(y)| < 1}$$

$\Longrightarrow$ the jump range of $\rho(X_t)$ is bounded

- $\rho \in \mathcal{A}$: fixed
- $v(R) = m(B_\rho(R))$
- $C_R = \dfrac{1}{32} \cdot \dfrac{R}{\log v(R) + \log \log R} \quad (R \geq 6)$

Assumption 1.

- $\exists\{\rho_R\} \subset \mathcal{A}$: **nondecreasing with** $\rho_1 = \rho$
- $\exists\{F_R\}$: **symmetric, nondecreasing and positive on** $X \times X$

(i) $\displaystyle \sup_{x\in X} \int_{d(x,y)\geq F_R(x,y)} J(x, \mathrm{d}y) < \infty$

(ii) $\exists R_0 > 0$ **s.t.** $\forall R \geq R_0$,

$$d(x, y) < F_R(x, y) \Longrightarrow |\rho_R(x) - \rho_R(y)| < C_R$$

(iii) $\forall K \subset X$; **compact,** $\exists R_1 \geq 1$ **s.t.** $\forall R \geq R_1$,

$$K \subset B_{\rho_R}(R/4)$$

▷ $M_1(R)$

$$= \sup_{x \in B_{\rho_R}(R)} \int_{d(x,y) < F_R(x,y)} (\rho_R(x) - \rho_R(y))^2 J(x, \mathrm{d}y)$$

▷ $M_2(R) = \sup_{x \in X} \int_{d(x,y) \geq F_R(x,y)} J(x, \mathrm{d}y)$

▷ $\psi(R) = \frac{R^2}{M_1(R)(\log v(R) + \log\log R)} \wedge \frac{1}{M_2(R)(\log R)^{1+\varepsilon}}$

$*$ cf. Grigor'yan ('99): $\psi(R) = \frac{R^2}{\log v(R)}$

Assumption 2.

(i) $\mathbb{M}$ **is conservative;**

(ii) $\psi(R) \nearrow \infty$ **as** $R \to \infty$**.**

Theorem. [S ('15+)]

Under Assumptions, $\exists c > 0$ **s.t. for q.e.** $x \in X$**,**

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le \psi^{-1}(ct) \text{ for all } t \ge T\right) = 1$$

Approach:

- L^2**-estimate of exit time distributions from balls**
- **Ikeda-Nagasawa-Watanabe's piecing out ('66)/**

  Meyer's construction ('75)

Barlow-Bass-Chen-Kassmann ('09), Grigor'yan-Hu-Lau ('14)

$$\mathcal{E}^R(u,u) = \iint_{d(x,y)<F_R(x,y)} (u(x)-u(y))^2 J(x,\mathrm{d}y)m(\mathrm{d}x)$$

▷ $\mathbb{M}^R = (\{X_t^R\}_{t\geq 0}, \{P_x\}_{x\in X})$: **associated with** $(\mathcal{E}^R, \mathcal{F})$

▷ $\tau^R_{B_{\rho_R}(R-C_R)} = \inf\left\{t>0 \mid X_t^R \notin B_{\rho_R}(R-C_R)\right\}$

$$\Longrightarrow X^R_{\tau^R_{B_{\rho_R}(R-C_R)}} \in B_{\rho_R}(R)$$

($\because$ jump range $=$ "$|\rho_R(x)-\rho_R(y)|$" $< C_R$)

▷ $u_R(t,x) = P_x\left(\tau^R_{B_{\rho_R}(R-C_R)} \leq t\right)$

Proposition 1.

For m-a.e. $x \in X$,

$$P_x\left(\tau_{B_{\rho_R}(R-C_R)} \le t\right) \le \underbrace{u_R(t,x)}_{\text{small jumps}} + \underbrace{tM_2(R)}_{\text{big jumps}}.$$

▷ $c(R) = 512 \cdot \dfrac{M_1(R)}{R^2} \cdot (\log v(R) + \log\log R)$

Proposition 2.

$\exists c > 0$ s.t. $\forall R \gg 1$, $\forall t > 0$,

$$\int_{B_{\rho_R}((R-C_R)/2)} u_R(t,x)^2\, m(\mathrm{d}x) \le \frac{c e^{c(R)t}}{v(R)^3 (\log R)^4}.$$

Example. $X = \mathbb{R}^d$, $J(x, \mathrm{d}y) = J(x, y)\mathrm{d}y$

(1) Assume that for some $\alpha \in (0, 2)$,

$$J(x, y) \lesssim \frac{1}{|x - y|^{d+\alpha}}$$

$\triangleright$ $R(t) = t^{\frac{1}{\alpha}}(\log t)^{1+\frac{1+\varepsilon}{\alpha}}$ $(\varepsilon > 0)$

$\Longrightarrow$ $P_x\left(\exists T > 0 \text{ s.t. } |X_t - x| \leq R(t) \text{ for all } t \geq T\right) = 1$

- $\rho_R(x) = |x|$
- $F_R(x, y) = C_R \left(= \frac{1}{32} \cdot \frac{R}{\log v(R) + \log\log R}\right)$

(2) Coefficient growth

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \mathrm{d}y$$

▷ $0 < \alpha < 2, \; 0 \leq q < \alpha$

$$c(x,y) \asymp \begin{cases} (1+|x|)^2 + (1+|y|)^2 & |x-y| < 1 \\ (1+|x|)^q + (1+|y|)^q & |x-y| \geq 1 \end{cases}$$

- $\rho_R(x) = \log(R + |x|)$
- $F_R(x,y) = c\{(R+|x|) \vee (R+|y|)\}$

(i) $0 \leq q < \alpha - 1$**:**

$\triangleright$ $R(t) = e^{ct}$ **(**$c > 0$**)**

$\Longrightarrow \exists c > 0$ **s.t.**

$$P_x\left(\exists T > 0 \text{ s.t. } |X_t - x| \leq R(t) \text{ for all } t \geq T\right) = 1$$

(ii) $\alpha - 1 \leq q < \alpha$**:**

$\triangleright$ $R(t) = \exp\left(ct^{\frac{1}{\alpha-q}}(\log t)^{\frac{1+\varepsilon}{\alpha-q}}\right)$ **(**$\varepsilon > 0, c > 0$**)**

$\Longrightarrow \forall \varepsilon > 0, \exists c > 0$ **s.t.**

$$P_x\left(\exists T > 0 \text{ s.t. } |X_t - x| \leq R(t) \text{ for all } t \geq T\right) = 1$$

Lower rate functions

▷ $\rho \in \mathcal{A}$: fixed

▷ $v(r) = m(B_\rho(r))$

▷ $w(R) = \operatorname*{ess\,sup}_{x \in X} \int_{X \setminus \{x\}} \{(\rho(x) - \rho(y))^2 \wedge R^2\} \, J(x, \mathrm{d}y)$

▷ $h(t) \asymp \dfrac{t^2}{w(t)}, \quad I(R) = \displaystyle\int_R^\infty \frac{h'(t)}{v(t)} \, \mathrm{d}t$

Example. transient symm. α-stable proc. on $\mathbb{R}^d$ (i.e., $d > \alpha$)

$$\rho(x) = |x| \Longrightarrow w(t) \asymp t^{2-\alpha}, \ \ h(t) \asymp t^\alpha$$

$$\Longrightarrow I(R) \asymp \frac{1}{R^{d-\alpha}}$$

Assumption.

(i) $\exists C > 0$ **s.t.** $v(2R) \leq Cv(R)$ **for any** $r > 0$;

(ii) $I(R) < \infty$ **for any** $R > 0$.

Theorem. [S ('15+)]

▷ $(\mathcal{E}, \mathcal{F})$: **regular transient Dirichlet form**

▷ $r(t)$: **strictly increasing positive function on** $(0, \infty)$ **s.t.**

$$\int_{\cdot}^{\infty} \frac{1}{I(r(s))} \sup_{x \in X} p_s(x,x)\, \mathrm{d}s < \infty \implies$$

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \geq r(t) \text{ for all } t \geq T\right) = 1, \text{ q.e.}$$

$\triangleright$ $e_K(x) = P_x(\sigma_K < \infty)$ (**$K \subset X$: compact**)

$\Longrightarrow \mathrm{Cap}(K) = \mathcal{E}(e_K, e_K) = \nu_K(K)$ (**ν_K: equilibrium pot.**)

Proposition.

(i) For any compact $K \subset X$ and $t > 0$,

$$P_x\,(\exists s > t \text{ s.t. } X_s \in K) \le \mathrm{Cap}(K) \int_t^\infty \sup_{y \in X} p_s(x, y)\,\mathrm{d}s.$$

(ii) $\exists C > 0$ s.t. for all $R > 0$,

$$\mathrm{Cap}(B_\rho(R)) \le \frac{C}{I(R)}.$$

(i) Bendikov-Saloff-Coste ('05) (ii) Ôkura ('03)

Example. $X = \mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$

$$\mathcal{E}(u,u) \asymp \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}} \frac{(u(x) - u(y))^2}{|x-y|^{d+\gamma(x,y)}} \mathrm{d}x\mathrm{d}y$$

▷ $\gamma(x,y)$: Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ s.t.

$$0 < \beta_1 \le \gamma(x,y) \le \beta_2 < 2, \quad |x-y| < 1$$

$$0 < \gamma_1 \le \gamma(x,y) \le \gamma_2 < 2, \quad |x-y| \ge 1$$

$\Longrightarrow p_t(x,x) \lesssim \dfrac{1}{t^{d/\gamma_2}}$ for any $t \ge 1$

[Barlow-Bass-Chen-Kassmann ('09), Chen-Kumagai ('08)]

Hence $(\mathcal{E}, \mathcal{F})$ is transient for $d > \gamma_2$.

∘ $w(t) \asymp t^{2-\gamma_1} \Longrightarrow I(R) \asymp \dfrac{1}{R^{d-\gamma_1}}$

$$\int_1^\infty \frac{1}{I(r(s))} p_s(x,x)\,\mathrm{d}s \lesssim \int_1^\infty \frac{r(t)^{d-\gamma_1}}{t^{d/\gamma_2}}\,\mathrm{d}t$$

▷ $r(t) = \dfrac{t^{\frac{1}{\gamma_2}\cdot\frac{d-\gamma_2}{d-\gamma_1}}}{(\log t)^{\frac{1+\varepsilon}{d-\gamma_1}}}$ $(\varepsilon > 0)$

$$\Longrightarrow \quad P_x\,(\exists T > 0 \text{ s.t. } |X_t - x| \geq r(t) \text{ for all } t \geq T) = 1$$

Remark. $\gamma_1 = \gamma_2$ $(= \gamma)$

$\Longrightarrow r(t) = \dfrac{t^{\frac{1}{\gamma}}}{(\log t)^{\frac{1+\varepsilon}{d-\gamma}}}$: lower rate ft. for symm. γ-stable proc.

3. Rate functions under heat kernel estimates

Joint work with Jian Wang (Fujian Normal Univ./Kyoto Univ.)

Consequence/Advantage

- ○ 0-1 laws for symm. jump processes with no scaling property
- ○ no use of the condition "$d(o,x) \in \mathcal{F}_{\rm loc}$"

($\Longrightarrow$ "symm. α-stable-like processes with $\alpha \geq 2$")

Related work: P. Kim-Kumagai-J. Wang ('15, arXiv)

LIL for $\sup_{0<s\leq t} d(X_s, X_0)$ and local times

Assumption.

$\exists p(t,x,y)$: **nonneg. symm. kernel on** $(0,\infty)\times X\times X$ **s.t.**

- $P_x(X_t \in \mathrm{d}y) = p(t,x,y)\, m(\mathrm{d}y)$
- $p(t+s,x,y) = \int_X p(t,x,z)p(s,z,y)\, m(\mathrm{d}z)$

Moreover, $\exists$**strictly inc. positive functions** V **and** ϕ **s.t.**

$$p(t,x,y) \asymp \frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(d(x,y))\phi(d(x,y))}$$

$$c_1\left(\frac{R}{r}\right)^{d_1} \le \frac{V(R)}{V(r)} \le c_2\left(\frac{R}{r}\right)^{d_2} \quad (0<r<R<\infty)$$

$$c_3\left(\frac{R}{r}\right)^{d_3} \le \frac{\phi(R)}{\phi(r)} \le c_4\left(\frac{R}{r}\right)^{d_4} \quad (0<r<R<\infty)$$

Remark. Under Assumption, $m(B_x(r)) \asymp V(r)$ **[KKW ('15)]**

Theorem 1 (with J. Wang).

- $d_1 > d_4$ **(**$\Rightarrow$ **transient)**
- $B_x(r)$ **is relatively compact for any** $x \in X$ **and** $r > 0$

$\triangleright$ $\varphi(t) = \phi^{-1}(t)g(t)$ **(**$g(t) \searrow 0$ **as** $t \to \infty$**)**

$$\int_1^\infty \frac{V(\varphi(t))}{\phi(\varphi(t))V(\phi^{-1}(t))}\,\mathrm{d}t < \infty \ (\text{or } = \infty)$$

$\Longrightarrow$ **for all** $x \in X$**,**

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \geq \varphi(t) \text{ for all } t \geq T\right) = 1 \ (\text{or } 0)$$

Theorem 2 (with J. Wang).

• $d_1 = d_2 = d_3 = d_4$ (⇒ recurrent and can not hit points)

▷ $\varphi(t) = \phi^{-1}(t)g(t)$ ($g(t) \searrow 0$ as $t \to \infty$)

$$\int_1^\infty \frac{1}{t|\log g(t)|}\,\mathrm{d}t < \infty \text{ (or } = \infty)$$

$\Longrightarrow$ for all $x \in X$,

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \geq \varphi(t) \text{ for all } t \geq T\right) = 1 \text{ (or } 0)$$

Example.

Assume that $\exists \alpha > 0$ **and** $\beta > 0$ **s.t.**

$$(*) \quad p(t,x,y) \asymp \frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x,y)^{\alpha+\beta}}$$

- $V(r) \asymp r^{\alpha}$ $(\Rightarrow d_1 = d_2 = \alpha)$
- $\phi(r) \asymp r^{\beta}$ $(\Rightarrow d_3 = d_4 = \beta)$

$$\mathcal{E}(u,u) = \iint_{X \times X \setminus \mathrm{diag}} (u(x) - u(y))^2 J(x,y) m(\mathrm{d}x) m(\mathrm{d}y)$$

$$\mathcal{F} = \left\{ u \in L^2(X;m) \mid \mathcal{E}(u,u) < \infty \right\}$$

Remark.

(i) $(*)$ **is valid if** $\exists \alpha > 0$ **and** $\beta \in (0,2)$ **s.t.**

$$m(B_x(r)) \asymp r^{\alpha}, \quad J(x,y) \asymp \frac{1}{d(x,y)^{\alpha+\beta}}$$

[Chen-Kumagai ('03, '08)]

(ii) $\beta \geq 2$ **for some subordinated diffusion processes**

[Kumagai ('03), Bogdan-Stós-Sztonyk ('03)]

$*$ **Ôkura ('02): subordinated symmetric Markov processes**

• $V(r) \asymp r^{\alpha}$, $\phi(r) \asymp r^{\beta}$

(i) $\alpha > \beta$ **(transient):**

$$\int_1^{\infty} \frac{V(\varphi(t))}{\phi(\varphi(t))V(\phi^{-1}(t))}\,\mathrm{d}t \asymp \int_1^{\infty} \frac{\varphi(t)^{\alpha-\beta}}{t^{\alpha/\beta}}\,\mathrm{d}t$$

$\triangleright$ $\varphi(t) = \dfrac{t^{\frac{1}{\beta}}}{(\log t)^{\frac{1+\varepsilon}{\alpha-\beta}}}$

$\varepsilon > 0$ $(\varepsilon \leq 0) \Longrightarrow$

$$P_x\,(\exists T > 0 \text{ s.t. } d(X_t, x) \geq \varphi(t) \text{ for all } t \geq T) = 1\ (= 0)$$

• $V(r) \asymp r^{\alpha}$, $\phi(r) \asymp r^{\beta}$

(ii) $\alpha = \beta$ **(recurrent and can not hit any point):**

$$\triangleright\ g(t) = \frac{1}{\phi^{-1}(t)t^{(\log\log t)^{1+\varepsilon}}} \quad (\varepsilon > -1)$$

$$\left(\Rightarrow \varphi(t) = \frac{1}{t^{(\log\log t)^{1+\varepsilon}}}\right)$$

$$\int_1^{\infty} \frac{1}{t|\log g(t)|}\,\mathrm{d}t \asymp \int_1^{\infty} \frac{1}{t(\log t)(\log\log t)^{1+\varepsilon}}\,\mathrm{d}t$$

$\varepsilon > 0$ $(-1 < \varepsilon \le 0) \Longrightarrow$

$$P_x\,(\exists T > 0 \text{ s.t. } d(X_t, x) \ge \varphi(t) \text{ for all } t \ge T) = 1\ (= 0)$$

(iii) Upper rate functions:

▷ $\varphi(t)$: increasing function

$$(**) \quad \int_1^\infty \frac{1}{\varphi(t)^\beta}\,\mathrm{d}t < \infty \ (= \infty) \implies$$

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \le \varphi(t) \text{ for all } t \ge T\right) = 1 \ (= 0)$$

▷ $\varphi(t) = t^{\frac{1}{\beta}}(\log t)^{\frac{1+\varepsilon}{\beta}}$

$$\boxed{(**) \iff \varepsilon > 0}$$

∗ **Generalized second Borel-Cantelli lemma**

[Chung-Erdős ('52), Takeuchi ('64)]

(i) $\sum_{k=1}^{\infty} P_x(C_k) = \infty$ **(ii)** $P_x(C_n \text{ i.o.}) = 0$ **or** 1

(iii) $\exists c > 0$ **s.t. for each fixed** $j \geq 1$**,**

$$P_x(C_i \cap C_j) \leq c P_x(C_i) P_x(C_j) \quad \text{for all } i \geq j + 2$$

$\Longrightarrow P_x(C_n \text{ i.o.}) = 1$