Upper escape rate of Markov chains on weighted graphs

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1. Introduction

▷ $(\{B_t\}_{t\geq 0}, P)$: **Brownian motion on** $\mathbb{R}^d$ **starting from** 0

Khintchine's LIL

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1 \quad P\text{-a.s.}$$

▷ $R_\varepsilon(t) := \sqrt{(2+\varepsilon)t\log\log t} \quad (\varepsilon > 0)$

$\Longrightarrow \forall\varepsilon > 0$,

$$P\left(|B_t| \leq R_\varepsilon(t) \text{ for all sufficiently large } t\right) = 1$$

Volume growth and global path properties

∘ **Symmetric diffusion processes**

∗ **Conservativeness**

Grigor'yan ('86), Takeda ('89, '92), Sturm ('94),...

∗ **Escape rate**

Grigor'yan ('99), Grigor'yan-Hsu ('08), Hsu-Qin ('10)

∘ **Markov chains on weighted graphs:**

∗ **Conservativeness**

Grigor'yan-Huang-Masamune ('12)

Folz (to appear in TAMS), Huang (to appear in POT)

∗ **Escape rate**

Huang (to appear in JOTP)

Purpose in this talk.

To obtain refined upper rate functions of Markov chains

2. Result

Weighted graph [Keller-Lenz ('11)]

▷ V: countably infinite set

▷ μ: $V \to (0, \infty)$

▷ w: $V \times V \to [0, \infty)$ s.t. $\forall x \in V$,

(i) $w(x, x) = 0$ (no loops)

(ii) $\displaystyle\sum_{z \in V} w(x, z) < \infty$ (finiteness of the "degree")

The triple (V, μ, w): **weighted graph**

○ $x \sim y \overset{\text{def}}{\iff} w(x,y) > 0$

○ $(x_0, \ldots, x_n)$: **path connecting x and y**

$\overset{\text{def}}{\iff} x_0 = x,\ x_n = y,\ x_{k-1} \sim x_k \ (k = 1, 2, 3, \ldots, n)$

<u>Assumption</u>

(V, μ, w): **locally finite and connected, i.e.**

- $\forall x \in V$, $\sharp\{y \in V \mid x \sim y\} < \infty$;
- $\forall x, y \in V \ (x \neq y)$, $\exists$**path connecting x and y**

▷ $Q = (q_{xy})_{x,y \in V}$: $V \times V$ matrix

$$q_{xy} = \begin{cases} \dfrac{w(x,y)}{\mu(x)} & x \neq y \\ -\dfrac{\sum_{z \in V} w(x,z)}{\mu(x)} & x = y \end{cases}$$

▷ $\mathrm{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in V})$: minimal Q-process

▷ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(V;\mu)$;

$$\mathcal{E}(u,u) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} w(x,y)(u(x) - u(y))^2$$

$$\mathcal{F} = \overline{C_0(V)}^{\sqrt{\mathcal{E}(\cdot,\cdot) + \|\cdot\|_{L^2(V;\mu)}}}$$

Adapted path metric

- **Huang-Keller-Masamune-Wojciechowski ('13)**

▷ $\boldsymbol{\sigma}$: $V \times V \to [0,1]$: **symmetric s.t.**

(i) $\boldsymbol{\sigma}(x,x) = 0, \;\; \forall x \in V$;

(ii) $\boldsymbol{\sigma}(x,y) > 0 \iff x \sim y$

○ $\boldsymbol{\sigma}$ **is adapted to** $(V, w, \mu) \;\; \overset{\text{def}}{\iff}$

$$\frac{1}{\mu(x)} \sum_{z \in V} w(x,z)\boldsymbol{\sigma}(x,z)^2 \leq 1, \;\; \forall x \in V$$

$$d_{\sigma}(x,y) := \inf\left\{ \sum_{k=1}^{n} \sigma(x_{k-1},x_k) \;\middle|\; \begin{array}{l} (x_0,\ldots,x_n)\text{: path} \\ \text{connecting } x \text{ and } y \end{array} \right\}$$

$$\Longrightarrow \frac{1}{\mu(x)} \sum_{z\in V} w(x,z) d_{\sigma}(x,z)^2 \le 1, \ \forall x \in V$$

$$\mathcal{E}(u,u) = \frac{1}{2} \sum_{x\in V} \left(\frac{1}{\mu(x)} \sum_{y\in V} w(x,y)(u(x)-u(y))^2 \right) \mu(x)$$

“$|\nabla d|^2(x) \le 1$” **and** “$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2(x)\,\mathrm{d}x$”

$$d_{\sigma}(x,y) := \inf\left\{ \sum_{k=1}^{n} \sigma(x_{k-1},x_k) \;\middle|\; \begin{array}{l} (x_0,\dots,x_n)\text{: path} \\ \text{connecting } x \text{ and } y \end{array} \right\}$$

$$\Longrightarrow \frac{1}{\mu(x)} \sum_{z\in V} w(x,z) d_{\sigma}(x,z)^2 \le 1, \ \forall x \in V$$

$$\mathcal{E}(u,u) = \frac{1}{2} \sum_{x\in V} \left(\frac{1}{\mu(x)} \sum_{y\in V} w(x,y)(u(x)-u(y))^2 \right) \mu(x)$$

$$\text{“}|\nabla d|^2(x) \le 1\text{”} \quad \text{and} \quad \text{“}\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2(x)\,\mathrm{d}x\text{”}$$

$$d_\sigma(x,y) := \inf\left\{ \sum_{k=1}^{n} \sigma(x_{k-1}, x_k) \;\middle|\; \begin{array}{l} (x_0,\ldots,x_n)\text{: path} \\ \text{connecting } x \text{ and } y \end{array} \right\}$$

$$\Longrightarrow \frac{1}{\mu(x)} \sum_{z\in V} w(x,z) d_\sigma(x,z)^2 \leq 1, \quad \forall x \in V$$

$$\mathcal{E}(u,u) = \frac{1}{2} \sum_{x\in V} \left(\frac{1}{\mu(x)} \sum_{y\in V} w(x,y)(u(x)-u(y))^2 \right) \mu(x)$$

"$|\nabla d|^2(x) \leq 1$" **and** "$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2(x)\,\mathrm{d}x$"

Intrinsic metric

- **Sturm ('94)**
- **Frank-Lenz-Wingert ('10)**

▷ $B_{d_\sigma}(x,r) := \{z \in V \mid d_\sigma(x,z) < r\}$ $(x \in V,\ \ r > 0)$

Condition.

- $\bar{x} \in V$: **reference point**

(i) $\inf_{x \in V} \mu(x) > 0$;

(ii) $\displaystyle\int_{\cdot}^{\infty} \frac{r}{\log \mu(B_{d_\sigma}(\bar{x}, r))}\,\mathrm{d}r = \infty$

Theorem (to appear in SPA).

Under Condition,

(1) M **is conservative;**

(2) $\exists c > 0$ **and** $\exists \hat{R} \geq 1$ **s.t. for**

$$\psi(R) := c \int_{\hat{R}}^{R} \frac{r}{\log \mu(B_{d_\sigma}(\bar{x}, r)) + \log \log r} \, \mathrm{d}r,$$

$$P_{\bar{x}}(d_\sigma(\bar{x}, X_t) \leq \psi^{-1}(t) \text{ for all sufficiently large } t) = 1$$

Condition.

- $\bar{x} \in V$: **reference point**

(i) $\inf_{x \in V} \mu(x) > 0$; **(ii)** $\int_{\cdot}^{\infty} \frac{r}{\log \mu(B_{d_\sigma}(\bar{x}, r))} \, \mathrm{d}r = \infty.$

Remark.

(i) Theorem (1) (conservativeness):

◦ **A weak version of Folz's (Condition (i))**

(ii) [Hsu-Qin ('10)]

$$\lim_{R\to\infty} \psi(R) = \infty \iff \textbf{Condition (ii)}$$

3. Approach

Modification of the weighted graph

- **Add vertices on edges $\Longrightarrow$ refined cut-off function**

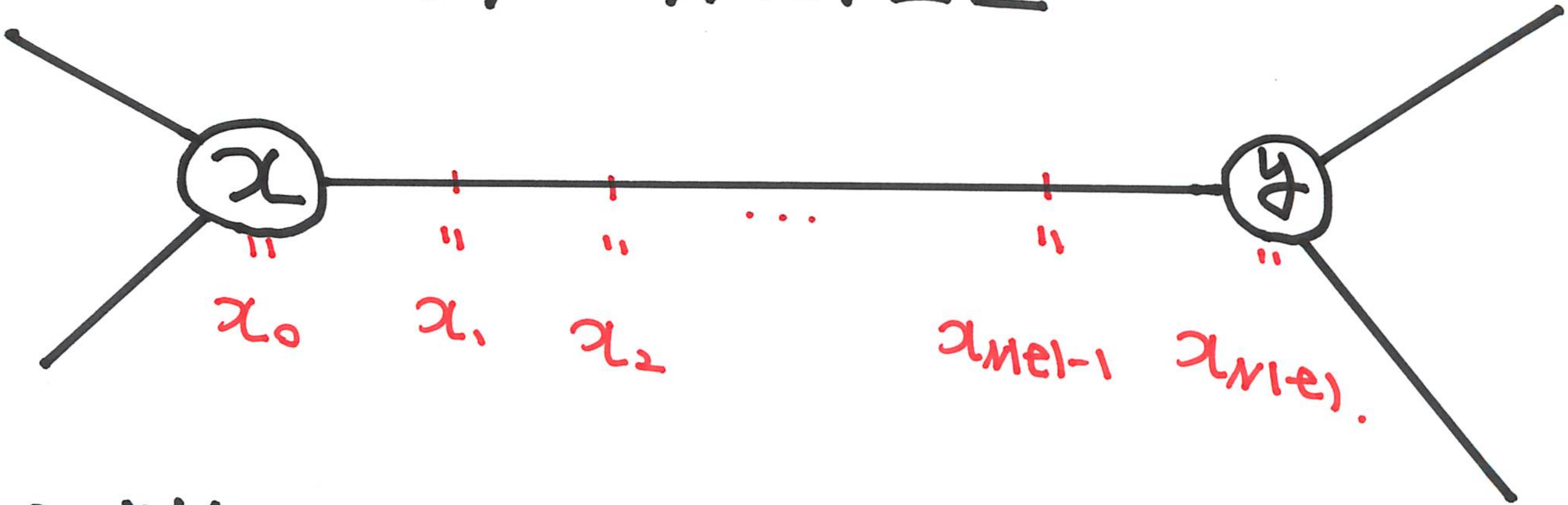
▷ (V_o, μ_o, w_o): **original weighted graph**

$$\Longleftrightarrow \quad (\{X_t^o\}_{t\geq 0}, \{P_x^o\}_{x\in V_o})$$

▷ (V, μ, w): **modified weighted graph**

$$\Longleftrightarrow \quad (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in V})$$

- $e = (x, y) \qquad N(e) \geq 2$



- $w(x_{k-1}, x_k) = N(e)\, w_0(x, y)$
- $\sigma(x_{k-1}, x_k) = \dfrac{\sigma_0(x, y)}{N(e)}$
- $\mu(x_k) = \begin{cases} \mu_0(x_k) & (x_k \in V_0) \\ \boxed{} & (x_k \in V \setminus V_0). \end{cases}$

$\Rightarrow \sigma$ is adapted to (V, w, μ)

- $A_t := \int_0^t 1_{V_o}(X_s)\,\mathrm{d}s$: total occupation time on V_o
- τ_t: right continuous inverse of A_t

Proposition.

(1) $(\{X_{\tau_t}\}_{t\geq 0}, \{P_x\}_{x\in V_o}) \overset{d}{=} (\{X_t^o\}_{t\geq 0}, \{P_x^o\}_{x\in V_o})$;

(2) If $(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in V})$ is conservative,

$\Longrightarrow \exists C > 1$, independent of (V_o, μ_o, w_o) and $\mathcal{N}$ s.t.

$$P_{\bar{x}}(\tau_t \leq Ct \quad \text{for all sufficiently large } t) = 1$$

▷ $R(t)$: **upper rate function of** (V, μ, w)

$\Longrightarrow R(Ct)$: **upper rate function of** (V_o, μ_o, μ_o)

$$
\begin{aligned}
d_{\sigma_o}(\bar{x}, X_t^o) &\overset{\text{Proposition}}{=} d_{\sigma_o}(\bar{x}, X_{\tau_t}) \\
&\overset{\text{definition}}{=} d_{\sigma}(\bar{x}, X_{\tau_t}) \\
&\overset{\text{upper rate ft}}{\leq} R(\tau_t) \\
&\overset{\text{Proposition}}{\leq} R(Ct)
\end{aligned}
$$

4. Example

Anti-trees

▷ x_0: **the origin**

▷ $\rho_0(x) := \rho(x_0, x)$: **graph distance between** x_0 **and** x

Assumption.

(1) $\mu(x) \equiv 1$, $w(x, y) = 1_{\{x \sim y\}}$;

(2) $\beta \in [0, 2]$

$$\deg(x) \asymp (\rho_0(x) + 2)^2 (\log(\rho_0(x) + 2))^\beta$$

$$\triangleright\ \sigma(x,y) := \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}} \wedge 1$$

$$\Longrightarrow \sigma(x,y) \asymp \frac{1}{(\rho_0(x)+2)(\log(\rho_0(x)+2))^{\beta/2}}$$

	$d_\sigma(x,y) \asymp$	$\log \mu(B_{d_\sigma}(x_0,r)) \asymp$
$0 \le \beta < 2$	$(\log(\rho_0(x)+2))^{(2-\beta)/2}$	$r^{2/(2-\beta)}$
$\beta = 2$	$\log\log(\rho_0(x)+3)$	e^{cr}

- **Condition (ii) holds** $\iff 0 \le \beta \le 1$

	$\psi^{-1}(t) \asymp$	$\rho_0(X_t) \lesssim$
$0 \le \beta < 1$	$t^{(2-\beta)/(2-2\beta)}$	$e^{t^{1/(1-\beta)}}$
$\beta = 1$	e^{ct}	$e^{c' e^{ct}}$

Remark: This result is sharp.

- **Conservativeness: Wojciechowski ('11), Folz**
- **Escape rate: Huang**