Escape rate of symmetric Markov processes

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1. 序

▷ $(\{B_t\}_{t\geq 0}, P)$: 原点から出発する d 次元ブラウン運動

Khintchine の重複対数の法則

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1 \quad P\text{-a.s.}$$

▷ $R_\varepsilon(t) := \sqrt{(2+\varepsilon)t\log\log t} \quad (\varepsilon > 0)$

$$\Longrightarrow \; P\left(|B_t| \leq R_\varepsilon(t) \text{ for all sufficiently large } t\right) = 1$$

▷ $0 < \alpha < 2$

▷ $(\{X_t\}_{t\geq 0}, P)$: 原点から出発する d 次元対称 α 安定過程

Khintchine ('38)

$$(*) \qquad \int_{\cdot}^{\infty} \frac{1}{R(t)^{\alpha}} \, \mathrm{d}t < \infty$$

$$\Longrightarrow P\left(|X_t| \leq R(t) \text{ for all sufficiently large } t\right) = 1$$

▷ $R_{\varepsilon}(t) := t^{\frac{1}{\alpha}}(\log t)^{\frac{1+\varepsilon}{\alpha}} \quad (\varepsilon > 0) \Longrightarrow (*)$

体積増大度と経路の大域的性質

○ 対称拡散過程

* 保存性

Gaffney ('59), 市原 ('86), Grigor'yan ('86), 竹田 ('89, '92),

Davies ('92), 大島 ('92), Sturm ('94)

$$\int_{\cdot}^{\infty} \frac{r}{\log m(B(r))} \mathrm{d}r = \infty \Longrightarrow \text{保存的}$$

(m: 対称化測度, $B(r)$: 半径 r の "球")

∗ **Upper escape rate**

Grigor'yan ('99), Grigor'yan-Hsu ('08), Hsu-Qin ('10), Ouyang ('13)

$\exists \hat{R} \geq 1, \exists c > 0$ s.t.

$$P_x\left(d(x, X_t) \leq \psi^{-1}(ct) \text{ for all sufficiently large } t\right) = 1$$

for

$$\psi(R) := \int_{\hat{R}}^{R} \frac{r}{\log m(B(r)) + \log\log r}\,\mathrm{d}r$$

○ **重み付きグラフ上のマルコフ連鎖**

∗ **保存性**

Grigor'yan-Huang-正宗 ('12), Folz ('13+), Huang ('13+)

∗ **Upper escape rate**

Huang ('13+), Huang-S. ('14)

○ 対称飛躍（拡散）型マルコフ過程

* 保存性

正宗-上村 ('11), Grigor'yan-Huang-正宗 ('12),

正宗-上村-J. Wang ('12), S.-上村 ('13+), S. ('13+)

本講演の興味.

Upper escape rate と体積増大度・係数増大度との関係

2. 結果

- ▷ (X, d): 局所コンパクト可分距離空間

- ▷ m: X 上の正のラドン測度 s.t. $\mathrm{supp}[m] = X$

- ▷ $(\mathcal{E}, \mathcal{F})$: $L^2(X; m)$ 上の正則ディリクレ形式 s.t.

$$\mathcal{E}(u, u) = \iint_{X \times X \setminus d} (u(x) - u(y))^2 \, J(x, \mathrm{d}y) m(\mathrm{d}x)$$

- ▷ $J(x, \mathrm{d}y)$: 飛躍核 s.t.

$$J(x, \mathrm{d}y) m(\mathrm{d}x) = J(y, \mathrm{d}x) m(\mathrm{d}y) \text{ on } X \times X \setminus d.$$

- ▷ $\mathbb{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in X})$: 対称ハント過程 ($\longleftrightarrow (\mathcal{E}, \mathcal{F})$)

例 (対称安定型過程). [Z.-Q. Chen-熊谷 ('03)]

$X = \mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$

▷ $c(x,y)$: $\mathbb{R}^d \times \mathbb{R}^d$ 上で可測かつ $c(x,y) \asymp 1$

▷ $J(x,\mathrm{d}y) = \dfrac{c(x,y)}{|x-y|^{d+\alpha}}\mathrm{d}y \quad (0 < \alpha < 2)$

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

注意. $c(x,y) = c_{d,\alpha} \Longrightarrow \mathcal{E}(u,u) = ((-\Delta)^{\alpha/2}u, u)_{L^2(\mathbb{R}^d)}$

$$\sup_{x \in X} \int_{X \setminus \{x\}} (1 \wedge d(x,y)^2) J(x, \mathrm{d}y) < \infty$$

<u>“適合した長さ”</u> [竹田 (’89), S. (’13+)]

▷ $\mathcal{A} := \left\{ \rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X) : \begin{array}{c} \lim_{x \to \Delta} \rho(x) = \infty \text{ かつ} \\ B_\rho(r) \text{ は相対コンパクト} \\ (\forall r > 0). \end{array} \right\}$

▷ $B_\rho(r) := \{x \in X : \rho(x) < r\}$

$\triangleright$ $F(x,y)$: $X \times X$ 上の対称な非負値関数

- $d(x,y) < F(x,y) \Longrightarrow |\rho(x) - \rho(y)| < 1$
- $\displaystyle \sup_{x \in X} \int_{d(x,y)<F(x,y)} (\rho(x) - \rho(y))^2 J(x, \mathrm{d}y) < \infty$
- $\displaystyle \sup_{x \in X} \int_{d(x,y)\geq F(x,y)} J(x, \mathrm{d}y) < \infty$

<u>内在的距離</u>　(“$|\nabla d| \leq 1$”)

- Sturm ('94), Frank-Lenz-Wingert ('10)
- Grigor'yan-Huang-正宗 ('12), Huang et. al. ('13)

例 [S. ('13+)].

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

▷ $\alpha \in (0,2), \ q \in [0,\alpha)$

$$c(x,y) \asymp \begin{cases} (1+|x|)^2 + (1+|y|)^2 & |x-y| < 1 \\ (1+|x|)^q + (1+|y|)^q & |x-y| \geq 1 \end{cases}$$

- $\rho(x) = \log(2+|x|) \in \mathcal{A}$
- $F(x,y) = \frac{1}{2}\{(1+|x|) \vee (1+|y|)\}$

問題. 長時間経った後の $\rho(X_t)$ の上限を特徴付ける:

$$P_x\left(\rho(X_t) \le R(t) \text{ for all sufficiently large } t\right) = 1.$$

▷ $R(t)$: $[0,\infty)$ 上の単調増加関数

小さな飛躍の積み重なり $>$ “ある程度” の大きな飛躍

▷ $v(R)$: 単調増加かつ $m(B_\rho(R)) \le v(R),\ \forall R > 0$

▷ $C_R := \dfrac{1}{32} \cdot \dfrac{R}{\log v(R) + \log\log R} \quad (R \ge 6)$

仮定 1.

○ $\exists\{\rho_R\} \subset \mathcal{A}$: **単調増加な非負値関数列**

○ $\exists\{F_R\}$: $X \times X$ **上の単調増加な非負値関数列**

(i) $\exists R_0 > 0$ **s.t.** $\forall R \geq R_0$,

$$d(x, y) < F_R(x, y) \Longrightarrow |\rho_R(x) - \rho_R(y)| \leq C_R$$

(ii) $\forall K \subset X$; **コンパクト**, $\exists R_1 \geq 1$ **s.t.** $\forall R \geq R_1$,

$$K \subset B_{\rho_R}(R/4)$$

▷ $C_R = \dfrac{1}{32} \cdot \dfrac{R}{\log v(R) + \log\log R} \quad (R \geq 6)$

- ▷ $M_1(R)$

$$:= \sup_{x \in B\rho_R(R)} \int_{d(x,y)<F_R(x,y)} (\rho_R(x) - \rho_R(y))^2 J(x, \mathrm{d}y)$$

- ▷ $M_2(R) := \sup_{x \in X} \int_{d(x,y) \geq F_R(x,y)} J(x, \mathrm{d}y)$

- ▷ $N_1(R)$: 単調増加かつ $M_1(R) \leq N_1(R)$
- ▷ $N_2(R)$: 単調減少かつ $M_2(R) \leq N_2(R)$

▷ $\psi_\mu(R) := \dfrac{R^{2-\mu}}{N_1(R)(\log v(R) + \log\log R)}$ $(0 \le \mu < 2)$

仮定 2.

(i) $\psi_\mu(R)$ **は単調増加かつ** $\lim_{R\to\infty} \psi_\mu(R) = \infty$;

(ii) $\exists c > 0$, $\exists \nu > 1$ **s.t.**

$$\psi_\mu(R) N_2(R) \le \frac{c}{(\log R)^\nu}, \qquad R \gg 1.$$

定理.

仮定 1, 2 の下, $\exists c > 0$ s.t. for m-a.e. $x \in X$,

$$P_x\left(\rho(X_t) \le \psi_\mu^{-1}(ct) \text{ for all sufficiently large } t\right) = 1.$$

系.

仮定 1, 2 の下, $\exists c > 0$ s.t. for m-a.e. $x \in X$,

$$\limsup_{t\to\infty} \frac{\rho(X_t)}{\psi_\mu^{-1}(ct)} \le 1, \quad P_x\text{-a.s.}$$

▷ $\psi_\mu(R) := \dfrac{R^{2-\mu}}{N_1(R)(\log v(R) + \log\log R)}$ $(0 \le \mu < 2)$

▷ $\psi_\mu(R)N_2(R) \le \dfrac{c}{(\log R)^\nu}, \qquad R \gg 1.$

3. 例.

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

$$\mathcal{F} = \overline{C_0^\infty(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}$$

▷ $\alpha \in (0,2)$, $\delta \in [0,1)$, $q \in [0,\alpha)$

- $|x-y| < 1$ のとき

$$c(x,y) \asymp (1+|x|)^2 \log(2+|x|)^{\delta} + (1+|y|)^2 \log(2+|y|)^{\delta}$$

- $|x-y| \geq 1$ のとき

$$c(x,y) \asymp (1+|x|)^{q} + (1+|y|)^{q}$$

(i) $\boldsymbol{\alpha - q > 1 - \delta}$**:**

$\exists \boldsymbol{c} > 0$ **s.t. for a.e.** $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{d}}$**,**

$$\limsup_{t\to\infty} \frac{|\boldsymbol{X_t} - \boldsymbol{X_0}|}{\exp\left(\boldsymbol{c}\boldsymbol{t}^{\frac{1}{1-\delta}}\right)} \leq 1, \quad \boldsymbol{P_x}\text{-a.s.}$$

(ii) $\boldsymbol{\alpha - q \leq 1 - \delta}$**:**

$\forall \varepsilon > 0$**: 十分小,** $\exists \boldsymbol{c} > 0$ **s.t. for a.e.** $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{d}}$**,**

$$\limsup_{t\to\infty} \frac{|\boldsymbol{X_t} - \boldsymbol{X_0}|}{\exp\left(\boldsymbol{c}\boldsymbol{t}^{\frac{1}{\alpha - q - \varepsilon}}\right)} \leq 1, \quad \boldsymbol{P_x}\text{-a.s.}$$

- $\rho_R(x) = \log(R + |x|) \in \mathcal{A}$
- $F_R(x,y) = C\{(R+|x|) \vee (R+|y|)\}$ ($C > 0$: 定数)

$$v(R) = c_0 e^{dR},\ N_1(R) = c_1 R^{\delta},\ N_2(R) = \frac{c_2}{R^{\alpha-q}}$$

- $\circ$ $\psi_\mu(R) = \dfrac{R^{2-\mu}}{N_1(R)(\log v(R) + \log\log R)} \asymp R^{1-\delta-\mu}$
- $\circ$ $\psi_\mu(R) N_2(R) \asymp \dfrac{1}{R^{\alpha-q-(1-\delta-\mu)}}$

注意. [S ('13+)] $\delta \le 1,\ q < \alpha \Longrightarrow (\mathcal{E}, \mathcal{F})$ は保存的

対称拡散過程との比較 [Ouyang ('13)]

▷ $\{a_{ij}(x)\}_{1\le i,j\le d}$: $\mathbb{R}^d$ 上の可測関数族 s.t.

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \asymp (1+|x|)^2\log(2+|x|)^\delta|\xi|^2, \quad \forall\xi\in\mathbb{R}^d$$

$$\mathcal{E}(u,u) = \sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\,\mathrm{d}x$$

$$\mathcal{F} = \overline{C_0^\infty(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}$$

(i) $0 \leq \delta < 1$ **のとき**

$$\limsup_{t\to\infty} \frac{|\boldsymbol{X}_t - \boldsymbol{X}_0|}{\exp\left(ct^{\frac{1}{1-\delta}}\right)} \leq 1, \quad P_x\text{-a.s.} \quad \text{a.e. } x \in \mathbb{R}^d$$

(ii) $\delta = 1$ **のとき**

$$\limsup_{t\to\infty} \frac{|\boldsymbol{X}_t - \boldsymbol{X}_0|}{\exp(\exp(ct))} \leq 1, \quad P_x\text{-a.s.} \quad \text{a.e. } x \in \mathbb{R}^d$$

注意. $\delta > 1 \Longrightarrow (\mathcal{E}, \mathcal{F})$ **は保存的とは限らない.**

4. 証明の方針.

$$P_x\left(\rho(X_t) \le \psi_\mu^{-1}(ct) \text{ for all sufficiently large } t\right) = 1.$$

▷ $A_n := \left\{\exists t \in (t_n, t_{n+1}] \text{ s.t. } \rho(X_t) \ge \psi_\mu^{-1}(ct)\right\}$

$$\sum_{n=\cdot}^{\infty} P_x(A_n) < \infty, \quad P_x\text{-a.s. } m\text{-a.e. } x \in X$$

$\Longrightarrow$ Borel-Cantelli の補題より

$$\rho(X_t) \le \psi_\mu^{-1}(ct) \quad \forall t \gg 1, \ P_x\text{-a.s. } m\text{-a.e. } x \in X$$

$$\mathcal{E}(u,u)=\iint_{X\times X\setminus d}(u(x)-u(y))^2\,J(x,\mathrm{d}y)m(\mathrm{d}x)$$

- $\triangleright$ $\mathbb{M}=(\{X_t\}_{t\geq 0},\{P_x\}_{x\in X})\longleftrightarrow(\mathcal{E},\mathcal{F})$

- $\triangleright$ $R_n=\theta^{n/2}\quad(1<\theta<2)$

- $\triangleright$ $t_n:=\dfrac{1}{c}\cdot\psi_\mu(R_n)\quad(c=1024)$

- $\triangleright$ $A_n:=\left\{\exists t\in(t_n,t_{n+1}]\text{ s.t. }\rho(X_t)\geq\psi_\mu^{-1}(ct)\right\}$

▷ $A_n := \left\{ \exists t \in (t_n, t_{n+1}] \text{ s.t. } \rho(X_t) \geq \psi_\mu^{-1}(ct) \right\}$

▷ $\tau_{B_{\rho_R}(R-C_R)} := \inf\{t > 0 \mid X_t \notin B_{\rho_R}(R - C_R)\}$

$\psi_\mu^{-1}(ct_n) = R_n \geq R_n - C_{R_n} \Longrightarrow$

$$P_x(A_n) \leq P_x\left(\sup_{0<t\leq t_{n+1}} \rho(X_t) \geq R_n - C_{R_n} \right)$$
$$= P_x\left(\tau_{B_{\rho_{R_n}}(R_n - C_{R_n})} \leq t_{n+1} \right)$$

▷ $R_n = \theta^{n/2} \quad (1 < \theta < 2)$ ▷ $t_n := \frac{1}{c} \cdot \psi_\mu(R_n)$

$$\mathcal{E}^R(u,u) = \iint_{d(x,y)<F_R(x,y)} (u(x)-u(y))^2 \, J(x,\mathrm{d}y)m(\mathrm{d}x)$$

▷ $\mathbb{M}^R = (\{X_t^R\}_{t\geq 0}, \{P_x\}_{x\in X}) \longleftrightarrow (\mathcal{E}^R, \mathcal{F})$

▷ $u_R(t,x) := P_x\left(\tau^R_{B_{\rho_R}(R-C_R)} \leq t\right)$

命題 1.

For m-a.e. $x \in X$,

$$P_x\left(\tau_{B_{\rho_R}(R-C_R)} \leq t\right) \leq u_R(t,x) + \underbrace{tN_2(R)}_{\text{大きい飛躍}}.$$

○ 池田-長澤-渡辺 ('66), Meyer ('75)

○ Grigor'yan-Hu-Lau ('13+)

$\triangleright$ $$I_R(t) := \int_{B_{\rho_R}(R-C_R)} e^{-\xi_R(t,x)} u_R(t,x)^2 \varphi_R(x)^2 \, m(\mathrm{d}x)$$

$\triangleright$ $$c_1(R) := 512 \cdot \frac{N_1(R)}{R^2} \cdot (\log v(R) + \log \log R)$$

$\triangleright$ $$c_2(R) := \frac{8}{R} \cdot (\log v(R) + \log \log R) \left(= \frac{1}{4C_R} \right)$$

$\triangleright$ $$\xi_R(t,x) := c_1(R)t + 2c_2(R)\rho_R(x)$$

$\triangleright$ $$\varphi_R(x) := \frac{\left(e^{c_2(R)(R-C_R)} - e^{c_2(R)\rho_R(x)} \right)_+}{e^{c_2(R)(R-C_R)} - 1}$$

命題 2.

$\exists c > 0$ **s.t.** $\forall R \gg 1, \forall t > 0,$

$$\int_{B_{\rho_R}((R-C_R)/2)} u_R(t,x)^2 \, m(\mathrm{d}x) \leq \frac{c e^{c_1(R)t}}{v(R)^3 (\log R)^4}.$$

▷ $R_n = \theta^{n/2} \quad (1 < \theta < 2)$

▷ $t_n := \dfrac{1}{c} \cdot \psi_\mu(R_n)$

$$\boxed{\int_{B_{\rho_{R_n}}((R_n - C_{R_n})/2)} u_{R_n}(t_{n+1}, x)^2 \, m(\mathrm{d}x) \leq \frac{c(\theta)}{v(R_n)^2 n^3}}$$

$$P_x(A_n) \leq P_x\left(\tau_{B_{\rho_{R_n}}(R_n - C_{R_n})} \leq t_{n+1}\right)$$

$$\leq u_R(t_{n+1}, x) + t_{n+1} N_2(R_n)$$

$K \subset X$: コンパクト

$$\int_K \left(\sum_{n=n_0(K)}^{\infty} P_x(A_n) \right) m(\mathrm{d}x)$$

$$\leq \sum_{n=n_0(K)}^{\infty} \int_K \left(u_{R_n}(t_{n+1}, x) + t_{n+1} N_2(R_n) \right) m(\mathrm{d}x)$$

$$< \infty$$

$$\boxed{t_{n+1} N_2(R_n) \lesssim \psi_\mu(R_n) N_2(R_n) \lesssim n^{-\nu}}$$