Conservation property of

symmetric jump-diffusion processes

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Nonlocal Operators:
Analysis, Probability, Geometry and Applications

ZiF Bielefeld
July, 2012.

1. Introduction

▷ X: **locally compact separable metric space**

▷ m: **positive Radon measure on X with full support**

$$\mathcal{E}(u,u) = \iint_{X\times X\setminus d} (u(x)-u(y))^2 J(x,\mathrm{d}y)m(\mathrm{d}x)$$

$$= (-\mathcal{L}u,u)_{L^2(X;m)}, \quad u \in \mathcal{F}\ (\subset L^2(X;m))$$

Typical example: $X=\mathbb{R}^d,\ m(\mathrm{d}x)=\mathrm{d}x,\ \alpha\in(0,2)$.

$$\mathcal{L} = -(-\Delta)^{\alpha/2} \iff J(x,\mathrm{d}y) = \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}}\,\mathrm{d}y$$

∘ $(\mathcal{E},\mathcal{F})$ is **conservative** (or **stochastically complete**)

$$\overset{\text{def}}{\iff} \; e^{\mathcal{L}t}1(x) = 1, \; m\text{-a.e. } x \in X, \; \forall t > 0. \qquad \textbf{(1)}$$

$$\text{“} \int_X p_t(x,y)\, m(\mathrm{d}y) = 1 \text{”}$$

▷ $\mathbb{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in X})$: m-symm. jump process on X

generated by $(\mathcal{E},\mathcal{F})$ (or $\mathcal{L}$)

$$\textbf{(1)} \iff P_x(X_t \in X) = 1, \; \text{q.e. } x \in X, \; \forall t > 0.$$

Diffusion case.

Gaffney, Ichihara, Grigor'yan, Takeda, Davies, Sturm,

Oshima, Hsu-Qin,...

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)\, m(\mathrm{d}x)$$

$$\mathcal{L}u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right)(x)$$

- **Coefficient growth** $(\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le c(1+|x|^2)|\xi|^2)$
- **Volume growth** $(m(\{x : |x| \le r\}) \le e^{cr^2})$

Return to Non-local case.

$$\mathcal{E}(u,u) = \iint_{X\times X\setminus d} (u(x)-u(y))^2\, J(x,\mathrm{d}y)m(\mathrm{d}x)$$

(i) Volume growth [Masamune-Uemura ('11), Grigor'yan-Huang-Masamune ('12), Masamune-Uemura-Wang ('12).]

- $\sup_{x\in X} \int_{X\setminus\{x\}} (1\wedge d(x,y)^2)\, J(x,\mathrm{d}y) < \infty$;
- $\exists x_0 \in X, \exists c > 0$ **s.t.**

$$m(B(x_0,r)) \le e^{c\, r\log r},\ \forall r > 0.$$

(ii) Coefficient growth [S.-Uemura ('12)].

Assume: $X = \mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$ (**essential**).

▷ $\gamma(x) \asymp |x| \quad (|x| \to \infty)$.

- $\exists M_1, M_2 > 0$ **s.t.**

$$\int_{|x-y|<\gamma(x)} |x-y|^2 \, J(x, \mathrm{d}y) \le M_1(1+|x|^2);$$

$$\int_{|x-y|\ge\gamma(x)} J(x, \mathrm{d}y) \le M_2;$$

"Condition on the drift part" ($\Leftarrow$ continuity of coefficient).

Consequence of the result in this talk.

(i) Improve [SU] (coefficient):

- **Remove "Condition on the drift part" ($\Rightarrow$ discontinuity)**
- **Underlying measure, state space: more general ($\Rightarrow$ degeneracy at the boundary)**

(ii) Generalize [MU], [GHM], [SU], [MUW]:

- **Balance between volume and coefficient**

2. Result.

▷ X: locally compact separable metric space

▷ m: positive Radon measure on X with full support

▷ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(X; m)$ s.t.

$$\mathcal{E}(u, u) = \iint_{X \times X \setminus d} (u(x) - u(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$

▷ $J(x, \mathrm{d}y)$: jump kernel s.t.

$$J(x, \mathrm{d}y)m(\mathrm{d}x) = J(y, \mathrm{d}x)m(\mathrm{d}y) \text{ on } X \times X \setminus d.$$

Assumption 1 ("big jumps").

$\exists F(x,y) > 0$: positive function on $X \times X \setminus d$ s.t.

(i) $F(x,y) = F(y,x)$;

(ii) $\displaystyle \sup_{x \in X} \int_{d(x,y) \geq F(x,y)} J(x,\mathrm{d}y) < \infty$.

$$\mathcal{E}^{(1)}(u,u) = \iint_{d(x,y) < F(x,y)} (u(x) - u(y))^2 \, J(x,\mathrm{d}y) m(\mathrm{d}x)$$

$$\Longrightarrow \quad \boxed{\mathcal{E}_1 \asymp \mathcal{E}_1^{(1)}} \quad \left(\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{L^2(X;m)} \right)$$

Lemma. $(\mathcal{E},\mathcal{F})$ is conservative iff so is $(\mathcal{E}^{(1)},\mathcal{F})$.

Takeda ('89) ("adapted distance")

$\triangleright$ $\mathcal{A} := \left\{ \rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X) : \begin{array}{l} \lim_{x \to \Delta} \rho(x) = \infty, \\ B_\rho(r) \text{ is compact}, \forall r > 0. \end{array} \right\}$

$\triangleright$ $B_\rho(r) := \{x \in X : \rho(x) \leq r\}$

$\triangleright$ $\mathbb{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in X})$: m-symm. jump process on X

generated by $(\mathcal{E}, \mathcal{F})$

Fix $\rho \in \mathcal{A}$. Then

$$\boxed{\mathbb{M} \text{ or } (\mathcal{E}, \mathcal{F}) \text{ is conservative} \iff \rho(X_t) < \infty, \forall t > 0.}$$

Assumption 2 ("small jumps").

$\exists \rho \in \mathcal{A}$ **s.t.**

(i) $\exists r > 0$ **s.t.**

$$|\rho(x) - \rho(y)| < r \quad \textbf{if} \quad d(x, y) < F(x, y);$$

(ii) $\sup_{x \in X} \int_{d(x,y) < F(x,y)} (\rho(x) - \rho(y))^2 \, J(x, \mathrm{d}y) < \infty.$

▷ $\Gamma^j(\rho)(x) := \int_{d(x,y)<F(x,y)} (\rho(x) - \rho(y))^2 J(x, \mathrm{d}y)$

▷ $M_\rho(r) := \operatorname{ess.sup}_{x \in B_\rho(r)} \Gamma^j(\rho)(x),\ r > 0$

Theorem.

If $\exists \{a_n\}$**: sequence s.t.**

$$\lim_{n\to\infty} e^{-na_n} M_\rho(n+3r) m(B_\rho(n+3r)) \cdot \exp\left(a_n^2 \exp\left(2a_n \sup_{\substack{\frac{n}{2}-r\le\rho(x)\le n+r,\\ d(x,y)<F(x,y)}} |\rho(x) - \rho(y)| \right) M_\rho(n+r) T \right) = 0$$

for some $T > 0$**, then** $(\mathcal{E}, \mathcal{F})$ **is conservative.**

3. Applications.

Example 1. (Coefficient growth.)

▷ $X = \mathbb{R}^d,\ \alpha \in (0,2),\ \beta > 0$

▷ $m(B(x,r)) \asymp r^{\beta},\ \forall x \in \mathbb{R}^d,\ \forall r > 0$

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}} m(\mathrm{d}x) m(\mathrm{d}y)$$

$$\mathcal{F} = \overline{C_0^\infty(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}$$

- $c(x,y) \asymp 1 \implies$ symmetric α-stable-like process

[Z.-Q. Chen-Kumagai ('03)]

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}} m(\mathrm{d}x) m(\mathrm{d}y)$$

- **For** $0 < |x-y| < 1$,

$$c(x,y) \asymp (1+|x|)^p + (1+|y|)^p \quad \text{for some} \quad p \le \alpha.$$

- **For** $|x-y| \ge 1$,

$$c(x,y) \asymp (1+|x|)^q + (1+|y|)^q \quad \text{for some} \quad q < \alpha.$$

○ $\rho(x) = \sqrt{\log(2+|x|)}, \ F(x,y) = \frac{1}{2}\{(1+|x|) \vee (1+|y|)\}$

Example 2. (Degeneracy at the boundary).

▷ $X = D \subset \mathbb{R}^d$: bounded smooth domain

▷ $\alpha \in (0, 2)$

$$\mathcal{E}(u, u) = \iint_{D \times D \setminus d} c(x, y) \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

$$\mathcal{F} = \overline{C_0^\infty(D)}^{\sqrt{\mathcal{E}_1}}$$

• $c(x, y) \equiv c \Longrightarrow$ censored stable process

[Bogdan-Burdzy-Z.-Q. Chen ('03)]

▷ $\delta_D(x) := d(x, \partial D)$, $x \in D$: **distance function**

- **For** $0 < |x-y| < (\delta_D(x) \vee \delta_D(y))/2$,

$$c(x,y) \asymp \delta_D(x)^p + \delta_D(y)^p \quad \text{for some} \quad p \geq \alpha.$$

- **For** $|x-y| \geq (\delta_D(x) \vee \delta_D(y))/2$,

$$c(x,y) \asymp \delta_D(x)^q + \delta_D(y)^q \quad \text{for some } q > \alpha.$$

Diffusion case ($\alpha = 2$). M.M.H. Pang ('88).

4. Sketch proof of Theorem.

▷ $\Gamma^j(\rho)(x) := \int_{d(x,y)<F(x,y)} (\rho(x) - \rho(y))^2 \, J(x, \mathrm{d}y)$

▷ $M_\rho(r) := \operatorname*{ess.sup}_{x \in B_\rho(r)} \Gamma^j(\rho)(x)$, $r > 0$

Theorem.

If $\exists \{a_n\}$: sequence s.t.

$$\lim_{n\to\infty} e^{-na_n} M_\rho(n+3r) m(B_\rho(n+3r)) \cdot \exp\left(a_n^2 \exp\left(2a_n \sup_{\substack{\frac{n}{2}-r\le\rho(x)\le n+r,\\ d(x,y)<F(x,y)}} |\rho(x) - \rho(y)| \right) M_\rho(n+r)T \right) = 0$$

for some $T > 0$, then $(\mathcal{E}, \mathcal{F})$ is conservative.

Approach: based on [GHM] (going back to Davies, Gaffney)

▷ $\{T_t\}_{t\geq 0}$: L^2-**semigroup associated with** $(\mathcal{E}^{(1)}, \mathcal{F})$

Lemma. **If** $\exists \{\varphi_n\} \subset \mathcal{F} \cap C_0(X)$ **with**

$$\lim_{n\to\infty} \varphi_n = 1 \quad m\text{-a.e.}$$

and $\exists t_0 > 0$ **s.t.** $\forall f \in \mathcal{F} \cap C_0(X)$,

$$\lim_{n\to\infty} \int_X (f(x) - T_t f(x))\varphi_n(x)\, m(\mathrm{d}x) = 0,\ \forall t \in (0, t_0)$$

$$\Longrightarrow \boxed{T_t 1 = 1 \ \ m\text{-a.e.},\ \forall t > 0.}$$

▷ $\{w_n\} \subset C_0^\infty(\mathbb{R})$ **s.t.**

(i) $w_n(t) = \begin{cases} 1 & |t| \leq n + r \\ 0 & |t| \geq n + 2r, \end{cases}$

(ii) $\exists L > 1$ **s.t.**

$$\sup_{n \geq 1} \sup_{t \in \mathbb{R}} |w_n'(t)| \leq L$$

▷ $\varphi_n(x) := w_n(\rho(x))(\in \mathcal{F} \cap C_0(X))$

$$\Longrightarrow \lim_{n \to \infty} \varphi_n = 1 \quad m\text{-}\mathbf{a.e.}$$

$\triangleright$ $u_t := T_t f \quad (f \in \mathcal{F} \cap C_0(X))$

$$\left(\int_X (f(x) - u_t(x))\varphi_n(x)\, m(\mathrm{d}x)\right)^2 = \left(\int_0^t \mathcal{E}^{(1)}(u_s, \varphi_n)\,\mathrm{d}s\right)^2$$

$$\leq 2tL^2\|f\|^2_{L^2(X;m)} \cdot e^{-na} M_\rho(n+3r) m(B_\rho(n+3r))$$

$$\cdot \exp\left(a^2 \exp\left(2a \sup_{\substack{\frac{n}{2}-r\leq\rho(x)\leq n+r \\ d(x,y)<F(x,y)}} |\rho(x)-\rho(y)|\right) M_\rho(n+r)t\right)$$

for any $a > 0$. **Take** $a = a_n$ **and** $n \to \infty$. $\square$