Conservation property of symmetric jump-diffusion processes

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Niigata University, Tokimate

March 16, 2012.

1. Introduction

▷ X: locally compact separable metric space

▷ m: positive Radon measure on X with full support

▷ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(X; m)$

⇔ $\mathrm{M} = (X_t, P_x)$: m-symmetric Hunt process on X

▷ $\zeta = \inf\{t > 0 : X_t \in \Delta\}$: life time of M

○ M is conservative $\underset{\text{def}}{\Longleftrightarrow}$ $P_x(\zeta = \infty) = 1$, q.e. $x \in X$.

Main factors.

◇ **Volume growth rate of** m

◇ **Growth rate of the associated coefficients**

Diffusion case.

Gaffney ('59), Ichihara ('86), Grigor'yan ('86), Takeda ('89), Davies ('92), Oshima ('92), Sturm ('94),...

Jump(-diffusion) case.

Masamune-Uemura ('11),

Grigor'yan-Huang-Masamune ('12?), S-Uemura ('11),

Masamune-Uemura-Wang ('12).

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} (u(x) - u(y))^2 \, J(x, \mathrm{d}y) m(\mathrm{d}x)$$

$$\mathcal{F} = \overline{C_0^{\mathrm{lip}}(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}$$

Conservativeness criteria.

(i) Volume growth rate [MU, GHM, MUW].

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d} (u(x)-u(y))^2 \, J(x,\mathrm{d}y)m(\mathrm{d}x)$$

- $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus\{x\}} (1\wedge|x-y|^2)\, J(x,\mathrm{d}y) < \infty$;
- $\exists c > 0$ s.t.

$$m(\{x\in\mathbb{R}^d : |x| \le r\}) \le e^{cr\log r}, \quad \forall r > 0.$$

(ii) Coefficient growth rate [SU].

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} (u(x) - u(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$

- $\exists M_1, M_2 > 0$ **s.t.**

$$\int_{|x-y| \le \gamma(x)} |x-y|^2 J(x, \mathrm{d}y) \le M_1(1 + |x|^2) \log(2 + |x|),$$

$$\int_{|x-y| \ge \gamma(x)} J(x, \mathrm{d}y) \le M_2 \left(\gamma(x) := \sqrt{4 + |x|^2}/2 \right),$$

"Condition on the drift part" ($\Longleftarrow$ continuity of coefficients);

- $m(\mathrm{d}x) = \mathrm{d}x$: **Lebesgue measure (essential).**

Purpose in this talk.

(i) Improve [SU]:

- Remove "Condition on the drift part"

($\Longrightarrow$ discontinuity of coefficients)

- Underlying measure, state space: more general

(ii) Generalize [MU], [GHM], [SU], [MUW]:

- Balance between volume and coefficients

2. Result.

$$\mathcal{E}(u,u) = \frac{1}{2}\mu^c_{\langle u\rangle}(X) + \iint_{X\times X\setminus d} (u(x)-u(y))^2 J(x,\mathrm{d}y)m(\mathrm{d}x)$$

- $\mu^c_{\langle u\rangle}$: positive Radon measure on X (energy measure)
- $J(x,\mathrm{d}y)$: jump kernel s.t.

$$J(x,\mathrm{d}y)m(\mathrm{d}x) = J(y,\mathrm{d}x)m(\mathrm{d}y) \text{ on } X\times X\setminus d.$$

Assumption 1 ("big jumps").

$\exists F(x,y) > 0$: **positive function on** $X \times X \setminus d$ **s.t.**

(i) $F(x,y) = F(y,x)$;

(ii) $\sup_{x \in X} \int_{d(x,y) \geq F(x,y)} J(x, \mathrm{d}y) < \infty$.

$$\mathcal{E}^{(1)}(u,u) := \frac{1}{2}\mu^c_{\langle u \rangle}(X) + \iint_{d(x,y) < F(x,y)} (u(x) - u(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$

$$\Longrightarrow \quad \boxed{\mathcal{E}_1 \asymp \mathcal{E}_1^{(1)}}$$

Lemma. $(\mathcal{E}, \mathcal{F})$ **is conservative iff so is** $(\mathcal{E}^{(1)}, \mathcal{F})$**.**

Takeda ('89)

▷ $\mathcal{F}_{\mathrm{loc,\,ac}} := \{\rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X) : \mu^c_{\langle\rho\rangle} \ll m\}$

▷ $B_\rho(r) := \{x \in X : \rho(x) \le r\}$

▷ $$\mathcal{A} := \left\{ \rho \in \mathcal{F}_{\mathrm{loc,\,ac}} : \begin{array}{l} \lim_{x\to\Delta} \rho(x) = \infty, \\ B_\rho(r) \text{ is compact, } \forall r > 0. \end{array} \right\}$$

Fix $\rho \in \mathcal{A}$.

$$\mathrm{M} \text{ is conservative} \iff \rho(X_t) < \infty, \ \forall t > 0$$

○ $Y_t := \rho(X_t)$: one-dimensional process

Diffusion case.

$$\begin{aligned}\mathcal{E}^Y(u,u) = \mathcal{E}(u \circ \rho, u \circ \rho) &= \frac{1}{2}\int_X u'(\rho(x))^2 \, \mu^c_{\langle \rho \rangle}(\mathrm{d}x) \\ &\leq \frac{1}{2}\int_X u'(\rho(x))^2 \, m(\mathrm{d}x) \\ &= \frac{1}{2}\int_{\mathbb{R}} u'(x)^2 \, (m \circ \rho^{-1})(\mathrm{d}x)\end{aligned}$$

Assumption 2 ("small jumps").

$\exists \rho \in \mathcal{A}$ **s.t.**

(i) $\exists r > 0$ **s.t.**

$$|\rho(x) - \rho(y)| < r \quad \textbf{if} \quad d(x,y) < F(x,y);$$

(ii) $\sup_{x \in X} \int_{d(x,y) < F(x,y)} (\rho(x) - \rho(y))^2 \, J(x, \mathrm{d}y) < \infty.$

- ▷ $\mu^{c}_{\langle\rho\rangle}(\mathrm{d}x) = \Gamma^{c}(\rho)(x)\, m(\mathrm{d}x)$
- ▷ $\Gamma^{j}(\rho)(x) := \int_{d(x,y)<F(x,y)} (\rho(x) - \rho(y))^2\, J(x, \mathrm{d}y)$
- ▷ $M_\rho(r) := \operatorname*{ess.\,sup}_{x\in B_\rho(r)} \Gamma^{c}(\rho)(x) + \operatorname*{ess.\,sup}_{x\in B_\rho(r)} \Gamma^{j}(\rho)(x),\ r > 0$

Theorem.

If $\exists\{a_n\}$: sequence s.t.

$$\lim_{n\to\infty} e^{-na_n} M_\rho(n+3r) m(B_\rho(n+3r)) \cdot \exp\left(a_n^2 \exp\left(2a_n \sup_{\substack{\frac{n}{2}-r\le\rho(x)\le n+r,\\ d(x,y)<F(x,y)}} |\rho(x)-\rho(y)| \right) M_\rho(n+r)T \right) = 0$$

for some $T > 0$, then $(\mathcal{E}, \mathcal{F})$ is conservative.

- $\mu^c_{\langle\rho\rangle}(\mathrm{d}x) = \Gamma^c(\rho)(x)\, m(\mathrm{d}x)$
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3. Applications.

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)\, m(\mathrm{d}x)$$
$$+ \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} (u(x) - u(y))^2\, J(x, \mathrm{d}y) m(\mathrm{d}x)$$

- **Assume** $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$**: closable.**

▷ $(\mathcal{E}, \mathcal{F})$**:** $\mathcal{E}_1$**-closure of** $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$

(i) Coefficient growth rates.

- $\exists \lambda > 0$ s.t.

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \lambda(1+|x|^2)\log(2+|x|)|\xi|^2, \ \forall \xi \in \mathbb{R}^d;$$

- $\exists M_1, M_2 > 0$ s.t. $\forall x \in \mathbb{R}^d$,

$$\int_{|x-y|<\frac{1+|x|}{2}} |x-y|^2 J(x,\mathrm{d}y) \le M_1(1+|x|^2)\log(2+|x|),$$

$$\int_{|x-y|\ge\frac{1+|x|}{2}} J(x,\mathrm{d}y) \le M_2;$$

- $\exists \beta > 0$ s.t. $m(B(r)) \le r^{\beta}, \ \forall r > 0.$

- $F(x,y) = \frac{1}{2}\{(1+|x|) \vee (1+|y|)\}$
- $\rho(x) = \sqrt{\log(2+|x|)}$

$\Longrightarrow \ |\rho(x) - \rho(y)| < 1, \forall x, \forall y$ **with** $|x-y| < F(x,y)$

- $a_n = 4\beta n$

Example.

$\triangleright$ $\alpha \in [1, 2)$, $\beta > 0$

$\triangleright$ $C_1 r^\beta \leq m(B_x(r)) \leq C_2 r^\beta, \quad \forall x \in \mathbb{R}^d, \forall r > 0$

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x) - u(y))^2}{|x-y|^{\alpha+\beta}} m(\mathrm{d}x) m(\mathrm{d}y)$$

• If $\exists c_1, c_2 > 0$ s.t. $c_1 \leq c(x,y) \leq c_2 \Longrightarrow$

symmetric α-stable-like process [Z.-Q. Chen-Kumagai ('03)]

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d} c(x,y)\frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}}\,m(\mathrm{d}x)m(\mathrm{d}y)$$

- **For** $0 < |x-y| < 1$,

$$c(x,y) \le c_1\{(1+|x|^2)\log(2+|x|)+(1+|y|^2)\log(2+|y|)\}.$$

- **For** $|x-y| \ge 1$,

$$c(x,y) \le c_2\{(1+|x|^2)^p + (1+|y|^2)^p\}$$

for some $p \in [0, \alpha/2)$

(ii) Volume and coefficient growth rates.

▷ $q \in (0,1]$: fixed

- $\exists \lambda > 0$ s.t.

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \lambda(1+|x|^2)^{1-q}|\xi|^2, \ \forall \xi \in \mathbb{R}^d;$$

- $\exists M_1, M_2 > 0$ s.t. $\forall x \in \mathbb{R}^d$,

$$\int_{|x-y|<(1+|x|)^{1-q}} |x-y|^2 J(x,\mathrm{d}y) \le M_1(1+|x|^2)^{1-q},$$

$$\int_{|x-y|\ge(1+|x|)^{1-q}} J(x,\mathrm{d}y) \le M_2;$$

- $\exists c > 0$ s.t. $m(B(r)) \le e^{cr^q \log r}, \ \forall r > 0.$

- $F(x,y) = \frac{c'}{2}\left\{(1+|x|)^{1-q} \vee (1+|y|)^{1-q}\right\}$

  ($c' > 0$: small enough)

- $\rho(x) = (1+|x|)^q$

$\Longrightarrow$ $|\rho(x) - \rho(y)| < c'$, $\forall x, \forall y$ **with** $|x-y| < F(x,y)$

- $a_n = p \log n$

Remark. $q = 1 \Longrightarrow$ **[GHM], [MUW]**

4. Sketch proof of Theorem.

Theorem.

If $\exists\{a_n\}$: sequence s.t.

$$\lim_{n\to\infty} e^{-na_n} M_\rho(n+3r) m(B_\rho(n+3r)) \cdot \exp\left(a_n^2 \exp\left(2a_n \sup_{\substack{\frac{n}{2}-r\le\rho(x)\le n+r,\\ d(x,y)<F(x,y)}} |\rho(x)-\rho(y)| \right) M_\rho(n+r)T \right) = 0$$

for some $T>0$, then $(\mathcal{E},\mathcal{F})$ is conservative.

- ▷ $\mu^c_{\langle\rho\rangle}(\mathrm{d}x) = \Gamma^c(\rho)(x)\, m(\mathrm{d}x)$
- ▷ $\Gamma^j(\rho)(x) := \int_{d(x,y)<F(x,y)} (\rho(x)-\rho(y))^2 J(x,\mathrm{d}y)$
- ▷ $M_\rho(r) := \operatorname*{ess.\,sup}_{x\in B_\rho(r)} \Gamma^c(\rho)(x) + \operatorname*{ess.\,sup}_{x\in B_\rho(r)} \Gamma^j(\rho)(x)$

▷ $\{T_t\}_{t\geq 0}$: L^2-semigroup associated with $(\mathcal{E}^{(1)}, \mathcal{F})$

Lemma. If $\exists\{\varphi_n\} \subset \mathcal{F} \cap C_0(X)$ with

$$\lim_{n\to\infty} \varphi_n = 1 \quad m\text{-a.e.}$$

and $\exists t_0 > 0$ s.t. $\forall f \in \mathcal{F} \cap C_0(X)$,

$$\lim_{n\to\infty} \int_X (f(x) - T_t f(x))\varphi_n(x)\, m(\mathrm{d}x) = 0,\ \forall t \in (0, t_0)$$

$$\Longrightarrow \boxed{T_t 1 = 1 \ \ m\text{-a.e.},\ \forall t > 0.}$$

▷ $\{w_n\} \subset C_0^\infty(\mathbb{R})$ **s.t.**

- $\circ$ $w_n(t) = \begin{cases} 1 & |t| \le n + r \\ 0 & |t| \ge n + 2r, \end{cases}$
- $\circ$ $\exists L > 1$ **s.t.**

$$\sup_{n\ge 1} \sup_{t\in\mathbb{R}} |w_n'(t)| \le L$$

▷ $\varphi_n(x) := w_n(\rho(x)) (\in \mathcal{F} \cap C_0(X))$

$$\Longrightarrow \lim_{n\to\infty} \varphi_n = 1 \quad m\text{-}\textbf{a.e.}$$

$\triangleright\ u_t := T_t f \quad (f \in \mathcal{F} \cap C_0(X))$

$$\left(\int_X (f(x) - u_t(x))\varphi_n(x)\, m(\mathrm{d}x)\right)^2 = \left(\int_0^t \mathcal{E}^{(1)}(u_s, \varphi_n)\,\mathrm{d}s\right)^2$$

$$\leq 2tL^2\|f\|^2_{L^2(X;m)} \cdot e^{-na} M_\rho(n+3r) m(B_\rho(n+3r))$$

$$\cdot \exp\left(a^2 \exp\left(2a \sup_{\substack{\frac{n}{2}-r\leq\rho(x)\leq n+r \\ d(x,y)<F(x,y)}} |\rho(x)-\rho(y)|\right) M_\rho(n+r)t\right)$$

for any $a > 0$. **Take** $a = a_n$ **and** $n \to \infty$. $\square$