三角関数のべき乗の積分

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1　三角関数のべき乗の積分

正弦（sin），余弦（sin）の積分

$$f_n(\theta) = \int \sin^n \theta d\theta \tag{1}$$

$$g_n(\theta) = \int \cos^n \theta d\theta \tag{2}$$

を求める（n は自然数）．

1.1　正弦

(i) $n = 1$ のとき,

$$f_1(\theta) = \int \sin \theta d\theta = -\cos\theta + C \tag{3}$$

(ii) $n = 2$ のとき,

$$\begin{aligned} f_2(\theta) &= \int \sin^2 \theta d\theta \\ &= \int \frac{1-\cos 2\theta}{2} d\theta \\ &= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C \end{aligned} \tag{4}$$

部分積分法を用いると,

$$\begin{aligned} f_{n+2}(\theta) &= \int \sin^{n+2} \theta d\theta \\ &= \int \sin^{n+1}\theta \cdot (-\cos\theta)' d\theta \\ &= \sin^{n+1}\theta \cdot (-\cos\theta) - \int (n+1)\sin^n\theta\cos\theta \cdot (-\cos\theta) d\theta \\ &= -\sin^{n+1}\theta\cos\theta + (n+1)\int \sin^n\theta(1-\sin^2\theta)d\theta \\ &= -\sin^{n+1}\theta\cos\theta + (n+1)\left[f_n(\theta) - f_{n+2}(\theta)\right] \end{aligned}$$

したがって,

$$f_{n+2}(\theta) = \frac{n+1}{n+2} f_n(\theta) - \frac{1}{n+2}\sin^{n+1}\theta\cos\theta \tag{5}$$

が成り立つ.

n が奇数のときは，(3), (5) 式を用いる．n が偶数のときは，(4), (5) 式を用いる．

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1.2　余弦

(i) $n = 1$ のとき，

$$g_1(\theta) = \int \cos\theta d\theta = \sin\theta + C \tag{6}$$

(ii) $n = 2$ のとき，

$$\begin{aligned} g_2(\theta) &= \int \cos^2\theta d\theta \\ &= \int \frac{1+\cos 2\theta}{2} d\theta \\ &= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C \end{aligned} \tag{7}$$

部分積分法を用いると，

$$\begin{aligned} g_{n+2}(\theta) &= \int \cos^{n+2}\theta d\theta \\ &= \int \cos^{n+1}\theta \cdot (\sin\theta)' d\theta \\ &= \cos^{n+1}\theta \cdot \sin\theta - \int (n+1)\cos^n\theta(-\sin\theta) \cdot \sin\theta d\theta \\ &= \cos^{n+1}\theta \sin\theta + (n+1)\int \cos^n\theta(1-\cos^2\theta) d\theta \\ &= \cos^{n+1}\theta \sin\theta + (n+1)\left[g_n(\theta) - g_{n+2}(\theta)\right] \end{aligned}$$

したがって，

$$g_{n+2}(\theta) = \frac{n+1}{n+2} g_n(\theta) + \frac{1}{n+2}\cos^{n+1}\theta \sin\theta \tag{8}$$

が成り立つ．

n が奇数のときは，(6), (8) 式を用いる．n が偶数のときは，(7), (8) 式を用いる．