A1　(5.9) 式の導出

Stirling の公式 (5.8) を用いて式 (5.5) を書き換えると

$$
\begin{aligned}
\text{(A1.1)} \qquad \log P_N(n) &= \log\left[\left(\frac{1}{2}\right)^N \frac{N!}{n!(N-n)!}\right] \\
&= -N\log 2 + \log N! - \log n! - \log(N-n)! \\
&\simeq -N\log 2 + \left[N\log N - N + \frac{1}{2}\log 2\pi N\right] - \left[n\log n - n + \frac{1}{2}\log 2\pi n\right] \\
&\quad -\left[(N-n)\log(N-n) - (N-n) + \frac{1}{2}\log[2\pi(N-n)]\right] \\
&= -N\log 2 + N\log N - N - n\log n + n - (N-n)\log(N-n) + (N-n) \\
&\quad +\frac{1}{2}\log 2\pi N - \frac{1}{2}\log 2\pi n - \frac{1}{2}\log[2\pi(N-n)] \\
&= -N\log 2 + N\log N - n\log n - (N-n)\log(N-n) \\
&\quad +\frac{1}{2}\log 2\pi N + \frac{1}{2}\log 2\pi N - \frac{1}{2}\log 2\pi N - \frac{1}{2}\log 2\pi n - \frac{1}{2}\log[2\pi(N-n)] \\
&= -N\log 2 + (N-n+n)\log N - n\log n - (N-n)\log(N-n) \\
&\quad -\frac{1}{2}\log 2\pi N - \left[\frac{1}{2}\log 2\pi n - \frac{1}{2}\log 2\pi N\right] - \left[\frac{1}{2}\log[2\pi(N-n)] - \frac{1}{2}\log 2\pi N\right] \\
&= -N\log 2 - n(\log n - \log N) - (N-n)[\log(N-n) - \log N] \\
&\quad -\frac{1}{2}\left[\log 2\pi N + \log\frac{2\pi n}{2\pi N} + \log\frac{2\pi(N-n)}{2\pi N}\right] \\
&= -N\left[\log 2 + \frac{n}{N}\log\frac{n}{N} + \frac{N-n}{N}\log\frac{N-n}{N}\right] \\
&\quad -\frac{1}{2}\left[\log 2\pi N + \log\frac{n}{N} + \log\frac{N-n}{N}\right] \\
&= -N\left[\log 2 + \frac{n}{N}\log\frac{n}{N} + \frac{N-n}{N}\log\frac{N-n}{N}\right] \\
&\quad -\frac{1}{2}\left[\log 2\pi N + \log\frac{n}{N} + \log\frac{N-n}{N}\right]
\end{aligned}
$$

ここで次のような変数を用いる

$$\text{(A1.2)} \qquad x = \frac{n}{N} - \frac{1}{2}, \quad |x| \ll 1$$

つまり

$$\text{(A1.3)} \qquad \frac{n}{N} = \frac{1}{2} + x = \frac{1}{2}(1+2x)$$

$$\text{(A1.4)} \qquad \frac{N-n}{N} = \frac{1}{2} - x = \frac{1}{2}(1-2x)$$

(A1.5)
$$\begin{aligned}
\log P_N(n) &\simeq -N\left[\log 2+\frac{n}{N}\log\frac{n}{N}+\frac{N-n}{N}\log\frac{N-n}{N}\right]\\
&\quad -\frac{1}{2}\left[\log 2\pi N+\log\frac{n}{N}+\log\frac{N-n}{N}\right]\\
&= -N\left[\log 2+\frac{1}{2}(1+2x)\log\frac{1}{2}(1+2x)+\frac{1}{2}(1-2x)\log\frac{1}{2}(1-2x)\right]\\
&\quad -\frac{1}{2}\left[\log 2\pi N+\log\frac{1}{2}(1+2x)+\log\frac{1}{2}(1-2x)\right]\\
&= -N\left[\log 2-\frac{1}{2}(1+2x)\log 2+\frac{1}{2}(1+2x)\log(1+2x)\right.\\
&\quad \left.-\frac{1}{2}(1-2x)\log 2+\frac{1}{2}(1-2x)\log(1-2x)\right]\\
&\quad -\frac{1}{2}[\log 2\pi N-\log 2+\log(1+2x)-\log 2+\log(1-2x)]\\
&= -N\left[\log 2-\frac{1}{2}(1+2x)\log 2-\frac{1}{2}(1-2x)\log 2\right.\\
&\quad \left.+\frac{1}{2}(1+2x)\log(1+2x)+\frac{1}{2}(1-2x)\log(1-2x)\right]\\
&\quad -\frac{1}{2}[\log 2\pi N-\log 2-\log 2+\log(1+2x)+\log(1-2x)]\\
&= -N\left[\frac{1}{2}(1+2x)\log(1+2x)+\frac{1}{2}(1-2x)\log(1-2x)\right]\\
&\quad -\frac{1}{2}\left[\log\frac{\pi N}{2}+\log(1+2x)+\log(1-2x)\right]
\end{aligned}$$

ここで，Taylor 展開を用いて次の近似をおく

(A1.6) $\log(1+t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3-\cdots$

従って

(A1.7) $\log(1+2x)\simeq 2x-2x^2$

(A1.8) $\log(1-2x)\simeq -2x-2x^2$

(A1.9)
$$\begin{aligned}
\log P_N(n) &\simeq -N\left[\frac{1}{2}(1+2x)\log(1+2x)+\frac{1}{2}(1-2x)\log(1-2x)\right]\\
&\quad -\frac{1}{2}\left[\log\frac{\pi N}{2}+\log(1+2x)+\log(1-2x)\right]\\
&\simeq -N\left[\frac{1}{2}(1+2x)(2x-2x^2)+\frac{1}{2}(1-2x)(-2x-2x^2)\right]\\
&\quad -\frac{1}{2}\left[\log\frac{\pi N}{2}+(2x-2x^2)+(-2x-2x^2)\right]\\
&\simeq -N(-2x^2+4x^2)-\frac{1}{2}\left[\log\frac{\pi N}{2}-4x^2\right]\\
&= -2(N-1)x^2-\frac{1}{2}\log\frac{\pi N}{2}\\
&\simeq -2Nx^2-\frac{1}{2}\log\frac{\pi N}{2}
\end{aligned}$$

書き直すと

(A1.10) $P_N(n)\simeq\sqrt{\frac{2}{\pi N}}e^{-2Nx^2}$