

A1 (4.9) 式の導出

Stirling の公式 (4.8) を用いて式 (4.5) を書き換えると

$$\begin{aligned}
 (A1.1) \quad \log P_N(n) &= \log \left[\left(\frac{1}{2} \right)^N \frac{N!}{n!(N-n)!} \right] \\
 &= -N \log 2 + \log N! - \log n! - \log(N-n)! \\
 &\simeq -N \log 2 + \left[N \log N - N + \frac{1}{2} \log 2\pi N \right] - \left[n \log n - n + \frac{1}{2} \log 2\pi n \right] \\
 &\quad - \left[(N-n) \log(N-n) - (N-n) + \frac{1}{2} \log[2\pi(N-n)] \right] \\
 &= -N \log 2 + N \log N - N - n \log n + n - (N-n) \log(N-n) + (N-n) \\
 &\quad + \frac{1}{2} \log 2\pi N - \frac{1}{2} \log 2\pi n - \frac{1}{2} \log[2\pi(N-n)] \\
 &= -N \log 2 + N \log N - n \log n - (N-n) \log(N-n) \\
 &\quad + \frac{1}{2} \log 2\pi N + \frac{1}{2} \log 2\pi n - \frac{1}{2} \log 2\pi N - \frac{1}{2} \log 2\pi n - \frac{1}{2} \log[2\pi(N-n)] \\
 &= -N \log 2 + (N-n+n) \log N - n \log n - (N-n) \log(N-n) \\
 &\quad - \frac{1}{2} \log 2\pi N - \left[\frac{1}{2} \log 2\pi n - \frac{1}{2} \log 2\pi N \right] - \left[\frac{1}{2} \log[2\pi(N-n)] - \frac{1}{2} \log 2\pi N \right] \\
 &= -N \log 2 - n(\log n - \log N) - (N-n)[\log(N-n) - \log N] \\
 &\quad - \frac{1}{2} \left[\log 2\pi N + \log \frac{2\pi n}{2\pi N} + \log \frac{2\pi(N-n)}{2\pi N} \right] \\
 &= -N \left[\log 2 + \frac{n}{N} \log \frac{n}{N} + \frac{N-n}{N} \log \frac{N-n}{N} \right] \\
 &\quad - \frac{1}{2} \left[\log 2\pi N + \log \frac{n}{N} + \log \frac{N-n}{N} \right] \\
 &= -N \left[\log 2 + \frac{n}{N} \log \frac{n}{N} + \frac{N-n}{N} \log \frac{N-n}{N} \right] \\
 &\quad - \frac{1}{2} \left[\log 2\pi N + \log \frac{n}{N} + \log \frac{N-n}{N} \right]
 \end{aligned}$$

ここで次のような変数を用いる

$$(A1.2) \quad x = \frac{n}{N} - \frac{1}{2}, \quad |x| \ll 1$$

つまり

$$(A1.3) \quad \frac{n}{N} = \frac{1}{2} + x = \frac{1}{2}(1+2x)$$

$$(A1.4) \quad \frac{N-n}{N} = \frac{1}{2} - x = \frac{1}{2}(1-2x)$$

$$\begin{aligned}
(A1.5) \quad \log P_N(n) &\simeq -N \left[\log 2 + \frac{n}{N} \log \frac{n}{N} + \frac{N-n}{N} \log \frac{N-n}{N} \right] \\
&\quad - \frac{1}{2} \left[\log 2\pi N + \log \frac{n}{N} + \log \frac{N-n}{N} \right] \\
&= -N \left[\log 2 + \frac{1}{2}(1+2x) \log \frac{1}{2}(1+2x) + \frac{1}{2}(1-2x) \log \frac{1}{2}(1-2x) \right] \\
&\quad - \frac{1}{2} \left[\log 2\pi N + \log \frac{1}{2}(1+2x) + \log \frac{1}{2}(1-2x) \right] \\
&= -N \left[\log 2 - \frac{1}{2}(1+2x) \log 2 + \frac{1}{2}(1+2x) \log(1+2x) \right. \\
&\quad \left. - \frac{1}{2}(1-2x) \log 2 + \frac{1}{2}(1-2x) \log(1-2x) \right] \\
&\quad - \frac{1}{2} [\log 2\pi N - \log 2 + \log(1+2x) - \log 2 + \log(1-2x)] \\
&= -N \left[\log 2 - \frac{1}{2}(1+2x) \log 2 - \frac{1}{2}(1-2x) \log 2 \right. \\
&\quad \left. + \frac{1}{2}(1+2x) \log(1+2x) + \frac{1}{2}(1-2x) \log(1-2x) \right] \\
&\quad - \frac{1}{2} [\log 2\pi N - \log 2 - \log 2 + \log(1+2x) + \log(1-2x)] \\
&= -N \left[\frac{1}{2}(1+2x) \log(1+2x) + \frac{1}{2}(1-2x) \log(1-2x) \right] \\
&\quad - \frac{1}{2} \left[\log \frac{\pi N}{2} + \log(1+2x) + \log(1-2x) \right]
\end{aligned}$$

ここで，Taylor 展開を用いて次の近似をおく

$$(A1.6) \quad \log(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \dots$$

従って

$$(A1.7) \quad \log(1+2x) \simeq 2x - 2x^2$$

$$(A1.8) \quad \log(1-2x) \simeq -2x - 2x^2$$

$$\begin{aligned}
(A1.9) \quad \log P_N(n) &\simeq -N \left[\frac{1}{2}(1+2x) \log(1+2x) + \frac{1}{2}(1-2x) \log(1-2x) \right] \\
&\quad - \frac{1}{2} \left[\log \frac{\pi N}{2} + \log(1+2x) + \log(1-2x) \right] \\
&\simeq -N \left[\frac{1}{2}(1+2x)(2x - 2x^2) + \frac{1}{2}(1-2x)(-2x - 2x^2) \right] \\
&\quad - \frac{1}{2} \left[\log \frac{\pi N}{2} + (2x - 2x^2) + (-2x - 2x^2) \right] \\
&\simeq -N(-2x^2 + 4x^2) - \frac{1}{2} \left[\log \frac{\pi N}{2} - 4x^2 \right] \\
&= -2(N-1)x^2 - \frac{1}{2} \log \frac{\pi N}{2} \\
&\simeq -2Nx^2 - \frac{1}{2} \log \frac{\pi N}{2}
\end{aligned}$$

書き直すと

$$(A1.10) \quad P_N(n) \simeq \sqrt{\frac{2}{\pi N}} e^{-2Nx^2}$$