

Failure of the index theorem in an incomplete market economy

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Abstract

Recently, it was proved that the index of an economy with incomplete real asset markets is typically $+1$ when the degree of incompleteness, which is defined as the difference between the number of states and the number of securities, is an even number. This paper considers the case where the degree of incompleteness is an odd number and proves that any odd number can be realized as the index of such an economy.

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1 Introduction

The index theorem was introduced to economics by Dierker (1972), who applied it to the Arrow-Debreu exchange economy. Although the index theorem is a mathematical concept, it has important economic implications pertaining to, for example, the existence of equilibrium, the number of equilibrium, and the local uniqueness of equilibrium. A unique feature of the index theorem is its detective power with respect to multiple equilibria. The index theorem insists that the sum of the indices at each equilibrium over all the equilibria of an economy, that is, the index of the economy, equals $+1$. Therefore, the existence of an equilibrium with index -1 implies that the economy has at least two other equilibria with index $+1$.

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The general equilibrium model with incomplete markets is an extension of the Arrow-Debreu model in that it describes the trading mechanism for uncertainty in a more precise manner. Therefore, a natural question to follow is whether or not the index theorem holds for the incomplete market model. Recent works by Momi (2003), Bich (2006), and Predtetchinski (2006) gave a partial answer to this question: the index theorem typically holds when the degree of incompleteness, which is defined as the difference between the number of states (S) and the number of securities (J), is an even number. The purpose of this paper is to study the case where the degree of incompleteness is an odd number.

The result of this paper is striking: it shows that any odd number can be the index of an incomplete market economy in which the degree of incompleteness is odd. That is, the index theorem does not hold for such an incomplete market economy. This is a sharp contrast to Dierker's index theorem for the Arrow-Debreu economy.

It is well known that the index is homotopy invariant. In fact, a simple proof of the index theorem for the Arrow-Debreu economy is obtained as a by-product of the homotopy path following for computation of an equilibrium, where the homotopy connects a single consumer's excess demand function to the aggregate excess demand function of the economy. Brown et al. (1996) and DeMarzo and Eaves (1996) presented homotopy methods to compute an equilibrium for the incomplete market model. A typical difficulty faced with the incomplete market model is discontinuity of demand functions at bad prices, where the budget set drops its rank. To overcome this difficulty, Brown et al. (1996) needed to switch the homotopies and Demarzo and Eaves (1996) employed a Grassmann manifold, which represents the space of income transfers between states. However these homotopy path followings do not provide the index theorem as simply as the Arrow-Debreu economy did, because homotopy paths should be switched in the former and Grassmann manifolds are not generally orientable in the latter.

Bich (2006) and Predtetchinski (2006) found that a homotopy is well defined on an orientable manifold when $S - J$ is even. In DeMarzo and Eaves (1996), the domain of the homotopy is the product of the price set and the Grassmann manifold (and the homotopy parameter set). However, when the asset structure is fixed, the entire Grassman manifold need not be considered and the domain can be restricted to a much smaller set, which is called the asset (pseudo) equilibrium manifold. Bich (2006) and Predtetchinski (2006) proved that the asset (pseudo) equilibrium manifold is orientable when $S - J$ is even, and hence, the index theorem holds.

Momi (2003) proved how the index change occurs when homotopies in Brown et al. (1996) are switched, that is, when a bad point where the budget set drops its rank is

passed through.¹ The index change does not occur at any bad points when $S - J$ is even, and hence, the index theorem holds. On the other hand, if $S - J$ is an odd number, the index change occurs at a bad point depending on how the homotopy path passes through the bad point; therefore such an economy might have an index not equal to $+1$.

This paper shows the existence of an economy whose index is not equal to $+1$ by conversely following the arguments of Momi (2003). Roughly speaking, when an odd number (not equal to $+1$) is given, we draw a picture of paths (passing through bad points) such that the given odd number is induced as the index of the picture when Momi's index change rule is applied. Next, we construct an economy that realizes the drawn paths as its homotopy paths. The economy will then have the given odd number as its index.

The main point of the paper is to construct an economy that realizes certain drawn paths as its homotopy paths. The following question is addressed: for arbitrarily drawn paths, is there an economy whose homotopy paths coincide with the drawn paths? This is the characterization problem of homotopy paths. The paper shows that any reasonable paths can be realized.

Note that the characterization of homotopy paths is closely related to the characterization of the equilibrium price set. In fact, they coincide at the terminal homotopy parameter value where the homotopy function equals the aggregate excess demand function. For the Arrow-Debreu economy, the characterization of the equilibrium price set is provided by Mas-Colell (1977). For the incomplete market economy, a similar result is recently obtained by Momi (2010). This result is applied to the construction of the economy in this paper. A drawback of the study by Momi (2010) is that the characterization is given for an asset structure where the budget set drops its rank in only one dimension. Therefore, the economy constructed in this paper to induce an arbitrarily given index number has such an asset structure.

The paper is organized as follows. Section 2 describes the setup of the incomplete market model. Section 3 reviews the approach in Momi (2003), on which this paper is based, and Section 4 states the main result of the paper. Section 5 summarizes the result of Momi (2010), which is applied to the construction of the economy. Sections 6 and 7 provide the proof of the paper's main theorem. Specifically, in Section 6, a picture of candidate homotopy paths is drawn which realizes an arbitrary given odd number as the index if the paths are taken as the homotopy paths of an economy. Section 7 shows that there exists an economy whose homotopy paths coincide with the drawn paths.

¹In fact, Momi (2003) switched the homotopies between the one in Brown et.al. (1996) and the one in DeMarzo and Eaves (1996).

2 The incomplete market model

We consider a standard two-period economy with incomplete real asset markets. There are S possible states in the second period and N goods in each state such that R^M , where $M = (S + 1)N$ with period 0 as state 0, represents the total commodity space. There are $J (\leq S)$ real assets A^j , $j = 1, \dots, J$, each of which promises the delivery of a bundle of commodities $A_s^j = (A_{s1}^j, \dots, A_{sN}^j)$ if state $s \in \{1, \dots, S\}$ occurs in the second period. We represent the asset structure by $A = \{A_{sn}^j\}$ and assume that the assets are unredundant. Each consumer indexed by i is defined by (\succsim^i, ω^i) where \succsim^i is the consumer's strictly convex, monotonic, continuous, and complete preference ordering on the consumption set R_+^M and $\omega^i \in R_{++}^M$ is the consumer's initial endowment vector. Let p , x^i , ω^i , and $z^i = x^i - \omega^i$ respectively denote price, consumption, endowment, and excess demand of consumer i , where $p = (p_0, \dots, p_S)$ and $p_s = (p_{s1}, \dots, p_{sN})$, and so on. We often relabel p as $p = (p_1, \dots, p_l, \dots, p_M)$, where $p_1 = p_{01}, p_2 = p_{02}, \dots, p_M = p_{SN}$ and relabel x^i , ω^i , and z^i in a similar manner.

The budget set, which the excess demand vector $z = (z_0, \dots, z_S) \in R^M$ satisfies, is

$$L(p) = \left\{ z \in R^M \left| \begin{array}{l} pz = 0, \\ p_1 \square z_1 \in \langle A(p) \rangle \end{array} \right. \right\},$$

where $p = (p_0, \dots, p_S) \in \Delta = \{p \in R_{++}^M \mid \|p\| = 1\}$ is the present value price system, $p_1 \square z_1$ denotes $[p_1 z_1, \dots, p_S z_S]^T$, and $\langle A(p) \rangle$ denotes the linear space in R^S spanned by J column vectors of the $S \times J$ payoff matrix $A(p) \equiv (p_s A_s^j)_{s=1, \dots, S}^{j=1, \dots, J}$.² See Duffie and Shafer (1985) for the justification behind this definition of a budget set using the present value price system. The excess demand function of consumer i is thus defined by

$$z^i(p) = \left\{ x - \omega^i \in R^M \left| \begin{array}{l} (x - \omega^i) \in L(p) \text{ and } x' \succsim^i x \text{ for any } x' \in R_+^M \\ \text{satisfying } (x' - \omega^i) \in L(p) \end{array} \right. \right\}.$$

The equilibrium price \bar{p} is defined by $\sum_i z^i(\bar{p}) = 0$. Thus, our incomplete market economy is defined by the set of consumers of a finite number and an asset structure $(\{\succsim^i, \omega^i\}_i, A)$.

Note that $L(p)$ is a $k \equiv M - (S - J) - 1$ -dimensional linear subspace in R^M if and only if the payoff matrix $A(p)$ is of full column rank. We label such a price as “good” and write Δ^g to denote the set of good prices $\Delta^g = \{p \in \Delta \mid \text{rank} A(p) = J\}$. We further label the critical price where the rank of $A(p)$ is less than J (that is, where the dimension of $L(p)$ is less than k) as “bad” and define $\Delta^b = \Delta \setminus \Delta^g = \{p \in \Delta \mid \text{rank} A(p) < J\}$. Since the budget set $L(p)$ drops its dimension at bad prices, the excess demand $z^i(p)$ is not continuous at bad prices.

² $\| \cdot \|$ denotes the Euclid norm and T denotes the transpose of a vector or a matrix.

Dealing with incomplete market models, it is sometimes convenient to consider one consumer as an unconstrained consumer who maximizes the utility subject to the complete market budget set $\{z \in R^M | pz = 0\}$. We consider the first consumer $i = 1$ as unconstrained and write her excess demand function as

$$z^u(p) = \left\{ x - \omega^1 \in R^M \left| \begin{array}{l} p(x - \omega^1) = 0 \text{ and } x' \succsim^i x \text{ for any } x' \in R_+^M \\ \text{satisfying } p(x' - \omega^1) = 0 \end{array} \right. \right\}$$

to distinguish it from her constrained excess demand $z^1(p)$. Let $Z^c = \sum_{i \geq 2} z^i$ denote the aggregate excess demand function of the remaining constrained consumers $i \geq 2$ and let $Z = z^u + Z^c$ denote the aggregate excess demand of the economy. See Duffie and Shafer (1985) for the justification and merits of considering an unconstrained consumer. In particular, $Z(\bar{p}) = 0$ implies $\sum_i z^i(\bar{p}) = 0$, and $\sum_i z^i(\bar{p}) = 0$ implies the existence of p' such that $Z(p') = 0$. Further, because of Walras' law, one of the M equations of $Z(p) = 0$ is redundant, and hence $\hat{Z}(\bar{p}) = 0$, where the symbol “ $\hat{\cdot}$ ” indicates that the M -th element is dropped, implies that \bar{p} is an equilibrium price.

Moreover, it is sometimes convenient to consider the excess demand defined for the k -dimensional linear subspaces of R^M , because $L(p)$ is such a space when $p \in \Delta^g$. We let $G^k(R^M)$ denote the set of k -dimensional linear subspaces of R^M , called the Grassmann manifold. We further define $G_{++}^k(R^M) = \{L \in G^k(R^M) | L \cap R_+^M = 0\}$. For a good price $p \in \Delta^g$, $L(p)$ is clearly an element of $G_{++}^k(R^M)$. We define

$$\tilde{z}^i(L) = \left\{ x - \omega^i \in R^M \left| \begin{array}{l} (x - \omega^i) \in L \text{ and } x' \succsim^i x \text{ for any } x' \in R_+^M \\ \text{satisfying } (x' - \omega^i) \in L \end{array} \right. \right\}$$

for $L \in G_{++}^k(R^M)$, and define $\tilde{Z}^c(L) = \sum_{i \geq 2} \tilde{z}^i$. Clearly, $\tilde{z}^i(L(p)) = z^i(p)$ for $p \in \Delta^g$.

As mentioned above, the function $p \mapsto L(p)$ is not continuous at $p \in \Delta^b$. To avoid this discontinuity, it is convenient to consider a pseudo equilibrium. Remember that $\langle A(p) \rangle$ is a J -dimensional linear subspace in R^S for $p \in \Delta^g$. For $p \in \Delta$ and $g \in G^J(R^S)$ where $G^J(R^S)$ denote the set of J -dimensional linear subspaces of R^S , we let $\mathcal{L}(p, g)$ denote the following budget set:

$$\mathcal{L}(p, g) = \left\{ z \in R^M \left| \begin{array}{l} pz = 0, \\ p_1 \square z_1 \in g \end{array} \right. \right\}.$$

Note that $\mathcal{L}(p, g)$ is always an element of $G_{++}^k(R^M)$ and the function $\mathcal{L} : \Delta \times G^J(R^S) \rightarrow G_{++}^k(R^M)$ is continuous. The pair $(\bar{p}, \bar{g}) \in \Delta \times G^J(R^S)$ is called a pseudo equilibrium when it satisfies $z^u(\bar{p}) + \tilde{Z}^c(\mathcal{L}(\bar{p}, \bar{g})) = 0$ and $\langle A(\bar{p}) \rangle \in \bar{g}$. If (\bar{p}, \bar{g}) is a pseudo equilibrium and $A(\bar{p})$ is of full column rank, then $\langle A(\bar{p}) \rangle = \bar{g}$ and \bar{p} is clearly an equilibrium price.

Subsequently, it is necessary to represent $g \in G^J(R^S)$ by local coordinate systems. We let Σ denote the set of permutations of $\{1, \dots, S\}$ and Π_σ be the permutation matrix such that $\Pi_\sigma \begin{bmatrix} X_1 \\ \vdots \\ X_S \end{bmatrix} = \begin{bmatrix} X_{\sigma(1)} \\ \vdots \\ X_{\sigma(S)} \end{bmatrix}$ for each $\sigma \in \Sigma$. By some $\sigma \in \Sigma$ and $Y = \begin{bmatrix} Y_{11} & \dots & Y_{1J} \\ \vdots & & \vdots \\ Y_{(S-J)1} & \dots & Y_{(S-J)J} \end{bmatrix} \in R^{(S-J)J}$, any $g \in G^J(R^S)$ can be written as

$$g = \{w \in R^S \mid [I|Y]\Pi_\sigma w = 0\}, \quad (1)$$

where I denotes the $(S - J) \times (S - J)$ identity matrix.³ For each $\sigma \in \Sigma$, define $W_\sigma = \{g \in G^J(R^S) \mid \text{there is } Y \in R^{(S-J)J} \text{ such that } g = \{w \in R^S \mid [I|Y]\Pi_\sigma w = 0\}\}$ and define $\varphi_\sigma : W_\sigma \rightarrow R^{(S-J)J}$ by $g = \{w \in R^S \mid [I|\varphi_\sigma(g)]\Pi_\sigma w = 0\}$. Then, $\{W_\sigma, \varphi_\sigma\}_{\sigma \in \Sigma}$ is actually an atlas for the Grassmann manifold $G^J(R^S)$. Refer to Duffie and Shafer (1985) and Momi (2003) for details of the above pseudo equilibrium and Grassmann manifold.

3 Result

We define the index at an equilibrium price $\bar{p} \in \hat{Z}^{-1}(0)$ by

$$\text{index} \hat{Z}(\bar{p}) = (-1)^{M-1} \text{sign} \left(\det \left[\frac{\partial \hat{Z}}{\partial \hat{p}}(\bar{p}) \right] \right),$$

where $\left[\frac{\partial \hat{Z}}{\partial \hat{p}} \right]$ denote the derivative of (Z_1, \dots, Z_{M-1}) with respect to (p_1, \dots, p_{M-1}) , that is, the Jacobian matrix of Z whose last row and last column are dropped.⁴ The index of the economy is defined by

$$\sum_{\bar{p} \in \hat{Z}^{-1}(0)} \text{index} \hat{Z}(\bar{p}).$$

This definition is equivalent to that in Momi (2003) regardless of the different price normalizations and is a natural extension of the index of the Arrow-Debreu economy. See Momi (2003) for the proof that the definition of the index is independent from price normalizations. The index of an economy is well defined only when $\hat{Z}^{-1}(0)$ is a finite set and the derivative of \hat{Z} is well defined at every $\bar{p} \in \hat{Z}^{-1}(0)$. The main result of this paper is given in the form of the following theorem.

Theorem. For any odd number, there exists an economy $(\{\preceq^i, \omega^i\}_i, A)$ with some numbers of states S , assets J , commodities N and consumers, whose index equals the odd number.

³We use I to denote identity matrices of all ranks. We do not mention the rank when it is clear. Moreover, we use 0 to denote any vector or matrix whose elements are 0 's.

⁴Note that $\text{sign}(a)$ denotes the sign of $a \in R$: $\text{sign}(a) = +1, 0, -1$ if $a > 0, a = 0, a < 0$, respectively; $\det[X]$ denotes the determinant of matrix $[X]$.

4 Homotopy, index, and index change rule

This section reviews Momi's (2003) approach to computing the index of the incomplete market economy and sketches the proof of the paper's main theorem. We define the homotopy $H : \Delta^g \times [0, 1] \rightarrow R^M$ between the unconstrained consumer's excess demand function $z^u(p)$ and the aggregate excess demand function $Z(p) = z^u(p) + Z^c(p)$ as

$$H(p, t) = z^u(p) + tZ^c(p).$$

Although we define H on $\Delta^g \times [0, 1]$, where H is continuous, it is convenient to work with $\overline{H^{-1}(0)}$, the closure of $H^{-1}(0)$ in $\Delta \times [0, 1]$.

Given below are the properties of $H^{-1}(0)$ that are satisfied for almost all initial endowments $\omega = \{\omega^i\}_i$ and asset structures A when each preference ordering \succsim^i is represented by a smooth utility function. These are well known as seen in Brown et al. (1996).

- (H1) $\overline{H^{-1}(0)}$ is a one-dimensional smooth compact submanifold of $\Delta \times [0, 1]$;
- (H2) $\overline{H^{-1}(0)}$ is bounded away from the boundary $\partial\Delta \times [0, 1]$;
- (H3) $\overline{H^{-1}(0)}$ is not tangent to the boundary $\Delta \times \{0, 1\}$;
- (H4) $(p^u, 0)$ is the unique intersection of $\overline{H^{-1}(0)}$ and the boundary $\Delta \times \{0\}$, where p^u is the supporting price of the indifference surface of the first consumer's preference ordering at ω^1 , and $p^u \in \Delta^g$;
- (H5) If $(p', 1)$ is the intersection of $\overline{H^{-1}(0)}$ and the boundary $\Delta \times \{1\}$, then $p' \in \Delta^g$;
- (H6) $\overline{H^{-1}(0)}$ intersects with $\Delta^b \times [0, 1]$ a finite number of times.

Typically, $H^{-1}(0)$ is drawn as in Figure 1. We call a one-dimensional connected manifold as a "path" and we call a path in $H^{-1}(0)$ or in $\overline{H^{-1}(0)}$ as a homotopy path. We further label each point (p, t) in $\overline{H^{-1}(0)}$ as a good point if p is a good price, and a bad point if p is a bad price.

We define the index of H at $(\bar{p}, \bar{t}) \in H^{-1}(0)$ as

$$index \hat{H}(\bar{p}, \bar{t}) = (-1)^{M-1} sign \left(det \left[\frac{\partial \hat{H}}{\partial \hat{p}}(\bar{p}, \bar{t}) \right] \right).$$

Note that because $Z(p) = H(p, 1)$, $Z(\bar{p}) = 0$ if and only if $H(\bar{p}, 1) = 0$ and $index \hat{Z}(\bar{p}) = index \hat{H}(\bar{p}, 1)$.

On a (connected) path in $H^{-1}(0)$, the index can be computed as done with the Arrow-Debreu model. See Mas-Colell et al. (1995) or Garcia and Zangwill (1981). When each preference \succsim^i is represented by a smooth utility function, for almost all initial endowments ω and asset structures A , following (R1) and (R2) hold.

(R1) The index of H at $(p^u, 0) \in H^{-1}(0)$ is $+1$.

(R2) \hat{H} is regular at any $(p, t) \in H^{-1}(0)$, that is, the $(M-1) \times M$ matrix $\left[\frac{\partial \hat{H}}{\partial \bar{p}}, \frac{\partial \hat{H}}{\partial t} \right]$ has rank $M-1$ at any $(p, t) \in H^{-1}(0)$.

(R2) then gives the index rule on $H^{-1}(0)$ which determines the relation between the indices at two different points on a path in $H^{-1}(0)$.

(R2') Following a homotopy path in $H^{-1}(0)$, the index of H is unchanged when moving along the same direction with respect to t but is changed when the direction is changed.

We say a path in $H^{-1}(0)$ is regular when the regularity of \hat{H} is satisfied at any point on the path.

The problem with the incomplete market economy is that $H^{-1}(0)$ does not necessarily provide a connected path that has both end points on the boundary $\Delta \times \{0, 1\}$. See Figure 1. The rule (R2') determines the relation between the index of H at (p', t') and the index of H at (p'', t'') . However, since $H^{-1}(0)$ is not connected, we cannot compute the index of H at (p''', t''') . To overcome this discontinuity, Momi (2003) utilized a pseudo path that was originally used by DeMarzo and Eaves (1996) to compute a pseudo equilibrium.

For $\sigma \in \Sigma$, we define the homotopy $K_\sigma = (K_\sigma^1, K_\sigma^2) : \Delta \times R^{(S-J)J} \times [0, 1] \rightarrow R^M \times R^{(S-J)J}$ by

$$K_\sigma^1(p, Y, t) = z^u(p) + t\tilde{Z}^c(\mathcal{L}(p, \varphi_\sigma^{-1}(Y))),$$

$$K_\sigma^2(p, Y) = [I|Y]\Pi_\sigma A(p).$$

Note that $(p, t) \in \overline{H^{-1}(0)}$ if and only if $(p, Y, t) \in K_\sigma^{-1}(0)$ for some $\sigma \in \Sigma$ and $Y \in R^{(S-J)J}$. In particular, $(p, t) \in H^{-1}(0)$ if and only if $(p, \varphi_\sigma(\langle A(p) \rangle), t) \in K_\sigma^{-1}(0)$ for σ such that $\langle A(p) \rangle \in W_\sigma$. In other words, the union of the projections of $K_\sigma^{-1}(0)$ onto $\Delta \times [0, 1]$ over $\sigma \in \Sigma$ coincides with $\overline{H^{-1}(0)}$. Therefore, we can follow the path in $K_\sigma^{-1}(0)$ instead of the path in $\overline{H^{-1}(0)}$.

We define the index of K_σ , regularity of \hat{K}_σ , and regular path in $K_\sigma^{-1}(0)$ analogously to those of H . The index of K_σ at $(\bar{p}, \bar{Y}, \bar{t}) \in K_\sigma^{-1}(0)$ is defined as

$$\text{index } \hat{K}_\sigma(\bar{p}, \bar{Y}, \bar{t}) = (-1)^{M-1+(S-J)J} \text{sign} \left(\det \begin{bmatrix} \frac{\partial \hat{K}_\sigma^1}{\partial \bar{p}}(\bar{p}, \bar{Y}, \bar{t}) & \frac{\partial \hat{K}_\sigma^1}{\partial \bar{Y}}(\bar{p}, \bar{Y}, \bar{t}) \\ \frac{\partial K_\sigma^2}{\partial \bar{p}}(\bar{p}, \bar{Y}, \bar{t}) & \frac{\partial K_\sigma^2}{\partial \bar{Y}}(\bar{p}, \bar{Y}, \bar{t}) \end{bmatrix} \right),$$

where $\frac{\partial \hat{K}_\sigma^1}{\partial Y}$ and $\frac{\partial K_\sigma^2}{\partial Y}$ respectively denote the derivatives of \hat{K}_σ^1 and K_σ^2 with respect to Y .⁵ The regularity of \hat{K}_σ at any $(p, Y, t) \in K_\sigma^{-1}(0)$, that is, that the $(M-1+(S-J)J) \times$

⁵Both K_σ^2 and Y are in the form of the matrix $(S-J) \times J$. To introduce an order in the elements of the matrix, we assume that the (i, j) element is before the (i', j') element when $i < i'$, or $i = i'$ and $j < j'$.

$(M - 1 + (S - J)J + 1)$ matrix $\begin{bmatrix} \frac{\partial \hat{K}_\sigma^1}{\partial \hat{p}} & \frac{\partial \hat{K}_\sigma^1}{\partial Y} & \frac{\partial \hat{K}_\sigma^1}{\partial t} \\ \frac{\partial \hat{K}_\sigma^2}{\partial \hat{p}} & \frac{\partial \hat{K}_\sigma^2}{\partial Y} & 0 \end{bmatrix}$ has rank $(M - 1 + (S - J)J)$ at any $(p, Y, t) \in K_\sigma^{-1}(0)$, is satisfied for almost all initial endowments w and asset structures A . We say that a path (one-dimensional connected manifold) in $K_\sigma^{-1}(0)$ is regular when the regularity of \hat{K}_σ is satisfied at any point on the path.

Momi (2003) proved the relation between $index \hat{H}$ at $(p, t) \in H^{-1}(0)$ and $index \hat{K}_\sigma$ at the corresponding point $(p, \varphi_\sigma(\langle A(p) \rangle), t)$.

Proposition 1. [Momi (2003)] Let $(p, t) \in H^{-1}(0)$ and $\sigma \in \Sigma$ satisfy $\langle A(p) \rangle \in W_\sigma$; then,

$$index \hat{H}(p, t) = (-1)^{(S-J)J} sign \left(\frac{1}{\left(\det [Q_\sigma(p)] \right)^{S-J}} \right) index \hat{K}(p, \varphi_\sigma(\langle A(p) \rangle), t), \quad (2)$$

where $Q_\sigma(p)$ is the $J \times J$ matrix consisting of the last J rows of $\Pi_\sigma A(p)$:

$$Q_\sigma(p) = \begin{bmatrix} p_{\sigma(S-J+1)} A_{\sigma(S-J+1)}^1 & \cdots & p_{\sigma(S-J+1)} A_{\sigma(S-J+1)}^J \\ \vdots & & \vdots \\ p_{\sigma(S)} A_{\sigma(S)}^1 & \cdots & p_{\sigma(S)} A_{\sigma(S)}^J \end{bmatrix}.$$

Note that $(p, t) \in H^{-1}(0)$ indicates that $A(p)$ has full column rank and that $\langle A(p) \rangle \in W_\sigma$ indicates that $Q_\sigma(p)$ has full rank.

Suppose that the regularities of \hat{H} and \hat{K} are satisfied in Figure 1. At (p'', t'') , $index \hat{K}_\sigma$ at $(p'', \varphi_\sigma(\langle A(p'') \rangle), t'')$ can be calculated from the $index \hat{H}(p'', t'')$ by Proposition 1. Then, the connected path in $K_\sigma^{-1}(0)$ can be followed from $(p'', \varphi_\sigma(\langle A(p'') \rangle), t'')$ to $(p''', \varphi_\sigma(\langle A(p''') \rangle), t''')$ counting the index of K_σ . The index of H at (p''', t''') can be calculated from the index of K_σ at $(p''', \varphi_\sigma(\langle A(p''') \rangle), t''')$ by Proposition 1. Thus we can compute the index of H at (p''', t''') evading the bad point. Note that we use the relation in Proposition 1 twice to evade one bad point.

If $S - J$ is even, then $(\det[Q_\sigma(p)])^{S-J}$ does not change its signs regardless of the sign of $\det[Q_\sigma(p)]$; hence, the index changes at bad points do not occur. That is, we can compute the index of H as if $\overline{H^{-1}(0)}$ yields standard homotopy paths. This is the proof given in Momi (2003) for the index theorem when $S - J$ is even.⁶

⁶Momi (2003) does not mention the regularity of \hat{K}_σ . As explained above, for the path following and index counting in $K_\sigma^{-1}(0)$, we need its regularity. Although the regularity of \hat{H} at a good point in $H^{-1}(0)$ implies the regularity of \hat{K}_σ at the corresponding point in $K_\sigma^{-1}(0)$, the regularity of \hat{K}_σ at the point corresponding to a bad point in $\overline{H^{-1}(0)}$ should also be mentioned. Since the regularity holds for almost all endowments and assets, the statement of the theorem in Momi (2003) is true: when $S - J$ is even, the index theorem holds for almost all endowments and assets.

On the other hand, if $S - J$ is odd, the index change at a bad point occurs when $\det[Q_\sigma(p)]$ changes its signs before and after the bad point. This suggests the possibility that an economy might have an index not equal to $+1$. Whether or not $\det[Q_\sigma(p)]$ changes its sign at a bad point depends on how the homotopy path in $\overline{H^{-1}(0)}$ passes through the bad point.

Now let us change our viewpoint. Suppose that we can arbitrarily draw a picture of paths in $\Delta^g \times [0, 1]$ (or in $\Delta \times [0, 1]$) and can determine the index number ($+1$ or -1) at one end point of each path. Then, applying Proposition 1 to count the indices, any odd number can be induced as the sum of the indices at the end points of the paths on the boundary $\Delta \times \{1\}$. The obvious problem to be considered is whether there is an economy $(\{\zeta^i, \omega^i\}_i, A)$ such that (i) the drawn paths in $\Delta^g \times [0, 1]$ are the actual regular homotopy paths of $H^{-1}(0)$, (ii) the corresponding paths in $\Delta \times R^{(S-J)J} \times [0, 1]$ are the actual regular homotopy paths of $K_\sigma^{-1}(0)$, and (iii) the given index number at one end point of each path is the actual index number. The existence of such an economy is shown as proof of this paper's theorem.

5 Characterization of equilibrium budget sets

This section reviews the result of Momi (2010), who proved that if the dimension of the budget set $L(p)$ is k or $k - 1$ for any price $p \in \Delta$, then any compact set of the budget sets can be the set of equilibrium budget sets of an incomplete market economy. At the final step of the proof of the paper's theorem, this result is applied to the construction of the economy.

Suppose an asset structure A is given. In Section 2, $\tilde{z}^i : G_{++}^k(R^M) \rightarrow R^M$ was defined as the excess demand when the consumer maximizes her utility subject to her net demand lying in $L \in G_{++}^k(R^M)$. Then the aggregate constrained excess demand function $\tilde{Z}^c = \sum_i \tilde{z}^i : G_{++}^k(R^M) \rightarrow R^M$ is continuous and satisfies the following:

(W) Walras' law: $\tilde{Z}^c(L) \in L$;

(BB) Boundedness from below: there is an $\alpha \in R$ such that $\tilde{Z}^c(L) > \alpha \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1) \in R^M$;

(BC) Boundary condition: if $L \rightarrow \bar{L} \in \partial G_{++}^k(R^M)$, then $\|\tilde{Z}^c(L)\| \rightarrow +\infty$,

because each function \tilde{z}^i satisfies these conditions. A question now arises: when a continuous function $z : G_{++}^k(R^M) \rightarrow R^M$ satisfying (W), (BB), and (BC) is given, is there a set of consumers $\{\zeta^i, \omega^i\}_i$ whose equilibrium budget sets coincide with $z^{-1}(0)$? This is an

open question that remains unanswered. However, answering this question is outside the scope of this paper because the whole set $G_{++}^k(R^M)$ is much larger than the set of budget sets arising from the fixed asst structure A . We are interested in the k -dimensional budget set $L(p)$ at good price $p \in \Delta^g$. To deal with all such budget sets, we define $G_{++} = \overline{L(\Delta^g)} \cap G_{++}^k(R^M) = \overline{L(\Delta^g)} \setminus \partial G_{++}^k(R^M)$ as the set of budget sets we need to consider.⁷

It is well known that the Grassmann manifold $G^k(R^M)$ is meterizable. We let $d(L, L')$ denote the distance between L and L' in $G^k(R^M)$ with respect to the introduced metric.⁸ For $\epsilon > 0$, we define

$$G_\epsilon = \{L \in G_{++} | d(L, \partial G_{++}^k(R^M)) > \epsilon\}$$

to denote the set of budget sets whose distance from the boundary $\partial G_{++}^k(R^M)$ is greater than ϵ . For any function $z : G_{++}^k(R^M) \rightarrow R^M$, we define

$$E_z = \{L \in G_{++} | z(L) = 0\}.$$

That is, E_z denotes the set of equilibrium budget sets in G_{++} when $z|_{G_{++}}$ is the aggregate excess demand function defined on G_{++} .

Proposition 2. [Momi (2010)] Let the dimension of $L(p)$ be k or $k - 1$ for any $p \in \Delta$. Let $z : G_{++}^k(R^M) \rightarrow R^M$ be a continuous function satisfying (W),(BB), and (BC). For any $\epsilon > 0$, there exist a positive $\mu < \epsilon$ and a finite set of consumers $\{\succsim^i, \omega^i\}_i$ whose aggregate excess demand function z^* , when they maximize their utilities subject to their net demands lying in $L - R_+^M$, satisfies $z^*|_{G_\mu} = z|_{G_\mu}$, $E_{z^*} = E_z \subset G_\mu$, and $\sum_{l=1}^M z_l^*(L) > 0$ for $L \in G_{++} \setminus G_\mu$.

It is an open question whether the condition that the dimension of $L(p)$ is k or $k - 1$ for any $p \in \Delta$ can be disregarded.

6 Candidate homotopy paths

This and next sections prove the paper's main theorem: for any odd number, there exists an economy $(\{\succsim^i, \omega^i\}_i, A)$ that has the odd number as its index.

As explained in Section 4, to obtain an index number not equal to $+1$, the following condition must be satisfied:

⁷For a function $x \mapsto f(x)$, we let $f(X) = \bigcup_{x \in X} f(x)$.

⁸We let $d(L, X) = \inf_{L' \in X} d(L, L')$ when X is a subset of $G^k(R^M)$.

(A1) $S - J$ is an odd number and $\Delta^b \neq \emptyset$.

Since we apply Proposition 2 to construct our economy, we need an asset structure such that

(A2) the dimension of the budget set $L(p)$ is k or $k - 1$ for any $p \in \Delta$.

For example, these conditions are satisfied if $S = 3$, $J = 2$, and the payoffs of assets are positive, $A_{sn}^j > 0$. Furthermore, from a technical reason, we select the asset structure such that

$$(A3) \quad A_s \equiv \begin{bmatrix} A_{s1}^1 & \cdots & A_{s1}^J \\ \vdots & & \vdots \\ A_{sN}^1 & \cdots & A_{sN}^J \end{bmatrix} \text{ is of full column rank, } s = 1, \dots, S.$$

As we will see later, this condition simplifies the discussion because it ensures the regularity of arbitrary paths.⁹

We next determine the first consumer (\bar{z}^1, ω^1) so that the supporting price p^u of her indifference surface at her initial endowment ω^1 is a good price and her excess demand function $z^u(p)$ is smooth.

Let α be an odd number. We wish to construct an economy whose index is α . The asset structure and the first consumer have been determined, and the constrained consumers $\{\bar{z}^i, \omega^i\}_{i \geq 2}$ now need to be determined. The construction proceeds as follows. In this section, a picture of paths is drawn to induce α as the index when it is computed by applying Proposition 1. In the next section, constrained consumers $\{\bar{z}^i, \omega^i\}_{i \geq 2}$ are constructed so that the drawn paths are the actual homotopy paths of the economy.

The candidate homotopy paths of $\overline{H^{-1}(0)}$ are represented as $(p^\lambda(r), t^\lambda(r))$, $\lambda = 1, \dots, \Lambda$, parameterized by $r \in [0, 1]$, where Λ is the number of the path we draw. Without loss of generality, we assume that $(\frac{\partial p^\lambda}{\partial r}(r), \frac{\partial t^\lambda}{\partial r}(r)) \neq (0, 0)$ at any r . The paths should satisfy (H1)–(H6) in Section 4, which are rewritten here as (h1)–(h6) respectively.

(h1) $p^\lambda : [0, 1] \rightarrow \Delta$ and $t^\lambda : [0, 1] \rightarrow [0, 1]$ are smooth functions;

(h2) $p^\lambda(r)$ are bounded away from the boundary $\partial\Delta$;

(h3) $\frac{\partial t^\lambda}{\partial r}(0) \neq 0$ and $\frac{\partial t^\lambda}{\partial r}(1) \neq 0$.

Without loss of generality, let one end point of the first path $\lambda = 1$ be $(p^u, 0)$.

⁹(A3) is used in Lemma 2 (i). The theorem can be proved without the condition. However, the candidate homotopy path should be chosen such that its regularity (regularity of $K_{\sigma F}^2$ at “bad” points) could be ensured later, that is, the path drawing and the construction of the economy cannot be separated.

(h4) $(p^1(0), t^1(0)) = (p^u, 0)$, which is the unique intersection of $(p^\lambda(r), t^\lambda(r))$ and $\Delta \times \{0\}$;

(h5) $(p^1(1), t^1(1)) = (p^1(1), 1)$ and $p^1(0) \in \Delta^g$; $(p^\lambda(0), t^\lambda(0)) = (p^\lambda(0), 1)$, $(p^\lambda(1), t^\lambda(1)) = (p^\lambda(1), 1)$ and $p^\lambda(0) \in \Delta^g$, $p^\lambda(1) \in \Delta^g$ for $\lambda \geq 2$; these are all intersections of $(p^\lambda(r), t^\lambda(r))$ and $\Delta \times \{1\}$;

(h6) the number of \bar{r} 's such that $p^\lambda(\bar{r}) \in \Delta^b$ is finite.

The paths $(p^\lambda(r), t^\lambda(r))$, $\lambda = 1, \dots, \Lambda$, in $\Delta \times [0, 1]$ are the candidate paths of $\overline{H^{-1}(0)}$. To indicate the corresponding paths in $\Delta \times R^{(S-J)J} \times [0, 1]$, which are candidate paths of $K_\sigma^{-1}(0)$, we write $g(p) \equiv \langle A(p) \rangle$ and define $g^\lambda : [0, 1] \rightarrow G^J(R^S)$ for each $\lambda = 1, \dots, \Lambda$, as

$$\begin{cases} g^\lambda(r) = g(p^\lambda(r)) \text{ for } r \text{ such that } p^\lambda(r) \in \Delta^g \\ g^\lambda(\bar{r}) = \lim_{r \rightarrow \bar{r}} g(p^\lambda(r)) \text{ for } \bar{r} \text{ such that } p^\lambda(\bar{r}) \in \Delta^b \end{cases} .$$

Note that g^λ is a smooth function, whereas $g(p^\lambda(r))$ is not continuous at \bar{r} such that $p^\lambda(\bar{r}) \in \Delta^b$. It is clear that $(p^\lambda(r), \varphi_\sigma(g^\lambda(r)), t^\lambda(r))$ corresponds to $(p^\lambda(r), t^\lambda(r))$ with the permutation σ satisfying $g^\lambda(r) \in W_\sigma$. Similarly, we define $L^\lambda : [0, 1] \rightarrow G_{++}$ as

$$\begin{cases} L^\lambda(r) = L(p^\lambda(r)) \text{ for } r \text{ such that } p^\lambda(r) \in \Delta^g \\ L^\lambda(\bar{r}) = \lim_{r \rightarrow \bar{r}} L(p^\lambda(r)) \text{ for } \bar{r} \text{ such that } p^\lambda(\bar{r}) \in \Delta^b \end{cases} .$$

Note that $L^\lambda(r)$ is a smooth function, whereas $L(p^\lambda(r))$ is not continuous at \bar{r} such that $p^\lambda(\bar{r}) \in \Delta^b$. It is clear that $L^\lambda(r) = \mathcal{L}(p^\lambda(r), g^\lambda(r))$ is the budget set corresponding to each point $(p^\lambda(r), t^\lambda(r))$.

For a smooth map $f : X \rightarrow Y$ between smooth manifolds, let $df_x : T_x X \rightarrow T_{f(x)} Y$ denote the derivative of f at $x \in X$, where $T_x X$ and $T_{f(x)} Y$ are the tangent spaces of X and Y at x and $f(x)$, respectively. For simplicity we often write $df : TX \rightarrow TY$. Furthermore when X is a (submanifold of) one dimensional Euclid space, we endow $T_x X$ the natural coordinate system and identify df_x with $df_x(1) \in T_{f(x)} Y$, the image of $1 \in T_x X$.

In addition to (h1)–(h6), we require our candidate paths to satisfy the following conditions.¹⁰

(h7) $z^u(p^\lambda(r)) \in L^\lambda(r)$ for any λ and r ;

(h8) $L^{\lambda'}(r') = L^{\lambda''}(r'')$ implies $(\lambda', r') = (\lambda'', r'')$;

(h9) $dL_r^\lambda \neq 0$ for any λ and r .

¹⁰We can prove that (h7)–(h9) are also typical properties of the homotopy paths satisfied for almost all endowments and asset structures in economies with smooth utility functions. We left the proof to readers. In this paper, we select the paths to satisfy these conditions.

To obtain an index not equal to $+1$, it is necessary for the paths to pass through bad points. For simplicity of argument, we will make the index changes occur at all bad points on our paths. Suppose that the index change occurs at every bad point in Figure 1. Then the possible index number of the picture is either $+1$ or -3 . While the index of the path with one end point on $\Delta \times \{0\}$ should be determined unambiguously, there are two possibilities for the index of each of the other paths that have both end points on $\Delta \times \{1\}$. In this sense, our paths should be consistent with α .

(P1) Under the assumption that index changes occur at any bad points, the paths $(p^\lambda(r), t^\lambda(r))$, $\lambda = 1, \dots, \Lambda$, are consistent with α .

To ensure the assumption in (P1), that is, for index changes occur at any bad points, the following condition should be satisfied:

(P2) If $p^\lambda(\bar{r}) \in \Delta^b$, then $\det[Q_\sigma(p^\lambda(r))]$ changes its sign at \bar{r} , where $\sigma \in \Sigma$ satisfies $\langle A(p^\lambda(r)) \rangle \in W_\sigma$ for r around \bar{r} .

Further, the regularity of the paths should be ensured as reviewed in Section 4. Note that however the regularity of the paths is not a condition on the paths themselves but rather a condition on the aggregate demand function of constrained consumers, which will be investigated in the next section. The following lemma, whose proof is given in Appendix, shows the existence of paths satisfying the conditions.

Lemma 1. There exist $(p^\lambda(r), t^\lambda(r))$, $r \in [0, 1]$, $\lambda = 1, \dots, \Lambda$, satisfying (h1)–(h9), (P1), and (P2).

7 Characterization of homotopy paths

In the previous section, we drew a picture of candidate paths that is consistent with index number α . The index number α is realized by suitably choosing index number ($+1$ or -1) at one end point of the paths $(p^\lambda(r), t^\lambda(r))$, $\lambda = 2, \dots, \Lambda$, and ensuring the regularity of the all paths. This section shows the existence of constrained consumers such that these are satisfied.

Proposition 3. Suppose that (\preceq^1, ω^1) is determined so that the consumer's unconstrained excess demand function z^u is smooth and $p^u \in \Delta^g$, the asset structure A satisfies (A1)–(A3), and the candidate paths $(p^\lambda(r), t^\lambda(r))$, $r \in [0, 1]$, $\lambda = 1, \dots, \Lambda$, satisfy (h1)–(h9). Moreover, the index number ($+1$ or -1) is given arbitrarily at the end point $(p^\lambda(0), 1)$ of each path $\lambda = 2, \dots, \Lambda$. Then, there exists a set of consumers $\{\preceq^i, \omega^i\}_{i \geq 2}$ such that

(i) $\overline{H^{-1}(0)}$ coincides with the union of the paths $(p^\lambda(r), t^\lambda(r))$, $r \in [0, 1]$, $\lambda = 1, \dots, \Lambda$;
(ii) if $p^\lambda(r) \in \Delta^g$, then \hat{H} is regular at $(p^\lambda(r), t^\lambda(r))$; (iii) if $p^\lambda(\bar{r}) \in \Delta^b$, then there exists $\bar{\sigma} \in \Sigma$ satisfying $g^\lambda(\bar{r}) \in W_{\bar{\sigma}}$ and $\hat{K}_{\bar{\sigma}}$ is regular at $(p^\lambda(r), \varphi_{\bar{\sigma}}(g^\lambda(r)), t^\lambda(r))$ for r in the neighborhood of \bar{r} ; and (iv) $index \hat{H}(p^\lambda(0), 1)$, $\lambda = 2, \dots, \Lambda$, coincide with the arbitrarily given index numbers at the corresponding points.

It is clear that Lemma 1 and this proposition establish the paper's theorem. The proof of this proposition proceeds as follows. First, a candidate aggregate excess demand function of constrained consumers is determined such that the statement of Proposition 3 is realized. Next, Proposition 2 shows the existence of a set of constrained consumers with this aggregate excess demand function.

7.1 Candidate aggregate excess demand function of constrained consumers

For a function $F : G_{++}^k(R^M) \rightarrow R^M$, we define $H_F : \Delta^g \times [0, 1] \rightarrow R^M$ as $H_F(p, t) = z^u(p) + tF(L(p))$ and define the index of H_F analogously to that of H . Similarly, we define $K_{\sigma F} : \Delta \times R^{(S-J)J} \times [0, 1] \rightarrow R^M \times R^{(S-J)J}$ as $K_{\sigma F} = (K_{\sigma F}^1, K_{\sigma F}^2)$, where $K_{\sigma F}^1(p, Y, t) = z^u(p) + tF(\mathcal{L}(p, \varphi_\sigma^{-1}(Y)))$.

In this subsection, we find a smooth function $F : G_{++}^k(R^M) \rightarrow R^M$ satisfying Walras' law: $F(L) \in L$, and the following conditions:

- (F1) $(p^\lambda(r), t^\lambda(r)) \in \overline{H_F^{-1}(0)}$, $r \in [0, 1]$, $\lambda = 1, \dots, \Lambda$;
- (F2) if $p^\lambda(r) \in \Delta^g$, then \hat{H}_F is regular at $(p^\lambda(r), t^\lambda(r))$;
- (F3) if $p^\lambda(\bar{r}) \in \Delta^b$, then there exists $\bar{\sigma} \in \Sigma$ satisfying $g^\lambda(\bar{r}) \in W_{\bar{\sigma}}$ and $\hat{K}_{\bar{\sigma} F}$ is regular at $(p^\lambda(r), \varphi_{\bar{\sigma}}(g^\lambda(r)), t^\lambda(r))$ for r in the neighborhood of \bar{r} ;
- (F4) the index of H_F at each $(p^\lambda(0), 1)$, $\lambda = 2, \dots, \Lambda$ equals the arbitrarily given index number (+1, or -1).

For obvious reasons, F is a candidate aggregate excess demand function of the constrained consumers. If F is indeed the aggregate excess demand function of the constrained consumers, then the economy satisfies (ii), (iii), and (iv) of Proposition 3.¹¹

To consider the regularities of \hat{H}_F and $\hat{K}_{\sigma F}$ together, we define $K_F^1(p, g, t) = z^u(p) + tF(\mathcal{L}(p, g))$ and $\tilde{K}_F^1(p, g, t) = \tilde{z}^1(\mathcal{L}(p, g)) + tF(\mathcal{L}(p, g))$ for $(p, g, t) \in \Delta \times G^J(R^S) \times [0, 1]$. We further define the asset (pseudo) equilibrium manifold as

$$V = \{(p, g) \in \Delta \times G^J(R^S) \mid \langle A(p) \rangle \subset g\}$$

¹¹At this stage, (F1) is slightly different from (i) in Proposition 3.

and consider the function $\tilde{K}_F^1|V \times [0, 1]$, the restriction of \tilde{K}_F^1 onto $V \times [0, 1]$. Note that any $(p, g) \in V$ satisfies $K_\sigma^2(p, g) = 0$ for any σ such that $g \in W_\sigma$. It is clear that our candidate paths satisfy $(p^\lambda(r), g^\lambda(r)) \in V$. See Bich (2006) and Predtetchinski (2006) for detailed investigations of the asset (pseudo) equilibrium manifold. In particular, note that it is an $M - 1$ -dimensional smooth manifold when (A3) is satisfied.¹²

The following lemmas show the relations between H_F , $K_{\sigma F}$, $K_F^1|V \times [0, 1]$, and $\tilde{K}_F^1|V \times [0, 1]$.

Lemma 2. Let $F : G_{++}^k(R^M) \rightarrow R^M$ be a smooth function satisfying $F(L) \in L$ and let $(p, g, t) \in V \times [0, 1]$ satisfy $K_F^1(p, g, t) = 0$.

(i) For any σ such that $g \in W_\sigma$, $K_{\sigma F}(p, \varphi_\sigma(g), t) = 0$. Furthermore, if $d(K_F^1|V \times [0, 1])_{(p,g,t)}$ is surjective onto p^\perp and if (A3) is satisfied, then $\hat{K}_{\sigma F}$ is regular at $(p, \varphi_\sigma(g), t)$.

(ii) If $p \in \Delta^g$, then $H_F(p, t) = 0$. Furthermore, if $dK_{F,(p,g,t)}^1$ is surjective onto p^\perp , then \hat{H}_F is regular at (p, t) . In particular, $dK_{F,(p,g,t)}^1(v, w, a) = dH_{F,(p,t)}(v, a)$ for $(v, w, a) \in T_{(p,g,t)}(V \times [0, 1])$.

Lemma 3. Let $F : G_{++}^k(R^M) \rightarrow R^M$ be a smooth function satisfying $F(L) \in L$ and let $(p, g, t) \in V \times [0, 1]$ satisfy $\tilde{K}_F^1(p, g, t) = K_F^1(p, g, t) = 0$. If $d(\tilde{K}_F^1|V \times [0, 1])_{(p,g,t)}$ is surjective onto $\mathcal{L}(p, g)$, then $d(K_F^1|V \times [0, 1])_{(p,g,t)}$ is surjective onto p^\perp .

See Appendix for the proofs of Lemmas 2 and 3. According to these lemmas, we should find F such that $\tilde{K}_F^1(p^\lambda(r), g^\lambda(r), t^\lambda(r)) = 0$ and $d(\tilde{K}_F^1|V \times [0, 1])_{(p^\lambda(r), g^\lambda(r), t^\lambda(r))}$ is surjective onto $\mathcal{L}(p^\lambda(r), g^\lambda(r))$.¹³

We proceed as follows. For each $\lambda = 1, \dots, \Lambda$, we choose $k - 1$ smooth maps $r \mapsto \alpha_l^\lambda(r)$, $l = 2, \dots, k$, such that $\alpha_l^\lambda(r) \in d\mathcal{L}_{(p^\lambda(r), g^\lambda(r))}(T_{(p^\lambda(r), g^\lambda(r))}V)$; and $dL_r^\lambda, \alpha_2^\lambda(r), \dots, \alpha_k^\lambda(r)$ are independent. See Claim 1 (vi) in Appendix for the existence of such maps.¹⁴

We also choose $k - 1$ smooth maps $r \mapsto \beta_l^\lambda(r)$, $l = 2, \dots, k$, such that $\beta_l^\lambda(r) \in L^\lambda(r)$; $z^u(p^\lambda(r)), \beta_2^\lambda(r), \dots, \beta_k^\lambda(r)$ are independent except at $(\lambda, r) = (1, 0)$ where $z^u(p^1(0)) = z^u(p^u) = 0$; and $\beta_l^1(0) = d\tilde{z}_{L^1(0)}^1(\alpha_l^1(0))$ for $(\lambda, r) = (1, 0)$, $l = 2, \dots, k$. The following lemma shows the existence of a desirable function F .

¹²For example, see Theorem 1 in Bich (2006).

¹³Because of (h7), the candidate paths satisfy $z^u(p^\lambda(r)) = \tilde{z}^1(\mathcal{L}(p^\lambda(r), g^\lambda(r)))$. Therefore $\tilde{K}_F^1(p^\lambda(r), g^\lambda(r), t^\lambda(r)) = 0$ implies $K_F^1(p^\lambda(r), g^\lambda(r), t^\lambda(r)) = 0$.

¹⁴For example, pick $(v_l^\lambda(r), w_l^\lambda(r)) \in T_{(p^\lambda(r), g^\lambda(r))}V$, $l = 2, \dots, k$, smooth with respect to r so that $v_l^\lambda(r) \in \mathcal{L}(p^\lambda(r), g^\lambda(r))$ and these $v_2^\lambda(r), \dots, v_k^\lambda(r)$ and the projection of dp_r^λ onto $\mathcal{L}(p^\lambda(r), g^\lambda(r))$ are independent on the k -dimensional space $\mathcal{L}(p^\lambda(r), g^\lambda(r))$. Then, $\alpha^\lambda(r)$ defined by $a_l^\lambda(r) = d\mathcal{L}_{(p^\lambda(r), g^\lambda(r))}(v_l^\lambda(r), w_l^\lambda(r))$, $l = 2, \dots, k$ satisfy the requirements.

Lemma 4. Suppose $\alpha_l^\lambda(r)$ and $\beta_l^\lambda(r)$, $l = 2, \dots, k$, are chosen as explained above. There exists a smooth function $F : G_{++}^k(R^M) \rightarrow R^M$ satisfying $F(L) \in L$ such that (i) $\tilde{z}^1(L^\lambda(r)) + t^\lambda(r)F(L^\lambda(r)) = 0$, and (ii) $(d\tilde{z}^1 + t^\lambda(r)dF)_{L^\lambda(r)} : T_{L^\lambda(r)}G_{++}^k(R^M) \rightarrow TR^M$ maps $\alpha_l^\lambda(r)$ to $\beta_l^\lambda(r)$, $l = 2, \dots, k$.

See Appendix for the proof of Lemma 4. It can be observed that F is a desirable function satisfying (F1)–(F4).

At the end point $r = 0$ of the first path $\lambda = 1$, $(p^1(0), t^1(0)) = (p^u, 0)$ satisfies (F1) and (F2) regardless of the function F . We consider the other points on the paths $(\lambda, r) \neq (1, 0)$.

First, Lemma 4 (i) implies $\tilde{K}_F^1(p^\lambda(r), g^\lambda(r), t^\lambda(r)) = 0$. Since $z^u(p^\lambda(r)) = \tilde{z}^1(\mathcal{L}(p^\lambda(r), g^\lambda(r)))$ on the paths, $K_F(p^\lambda(r), g^\lambda(r), t^\lambda(r)) = 0$ also holds.

Next, from the choice of $\alpha_l^\lambda(r)$, there exist $(v_l^\lambda(r), w_l^\lambda(r)) \in T_{(p^\lambda(r), g^\lambda(r))}V$ such that $\alpha_l^\lambda(r) = d\mathcal{L}_{(p^\lambda(r), g^\lambda(r))}(v_l^\lambda(r), w_l^\lambda(r))$, and Lemma 4 (ii) implies that $d(\tilde{K}_F|V \times [0, 1])_{(p^\lambda(r), g^\lambda(r), t^\lambda(r))}$ maps $(v_l^\lambda(r), w_l^\lambda(r), 0) \in T_{(p^\lambda(r), g^\lambda(r), t^\lambda(r))}(V \times [0, 1])$ to $\beta_l^\lambda(r)$, $l = 2, \dots, k$. It is clear that it maps $(0, 0, 1) \in T_{(p^\lambda(r), g^\lambda(r), t^\lambda(r))}(V \times [0, 1])$ to $F(L^\lambda(r))$, which is parallel to $z^u(p^\lambda(r))$ when (i) holds. Since $\beta_l^\lambda(r)$, $l = 2, \dots, k$, and $z^u(p^\lambda(r))$ are independent vectors in the k -dimensional space $L^\lambda(r)$, $d(\tilde{K}_F|V \times [0, 1])_{(p^\lambda(r), g^\lambda(r), t^\lambda(r))}$ is surjective onto $L^\lambda(r)$. Then, Lemmas 2 and 3 ensure that F satisfies (F1)–(F3).

We have not yet considered (F4). Select any $\lambda = 2, \dots, \Lambda$. (F4) is achieved by suitably choosing $\beta_l^\lambda(r)$ or $\alpha_l^\lambda(r)$, $l = 2, \dots, k$. For example, take $-\beta_k^\lambda(r)$ in the place of $\beta_k^\lambda(r)$, leaving the other $\beta_l^\lambda(r)$, $l = 2, \dots, k - 1$, and $\alpha_l^\lambda(r)$, $l = 2, \dots, k$, unchanged, and construct a new function F . Then the index at $(p^\lambda(r), t^\lambda(r))$ changes its sign. A detailed discussion follows.

Recall that the index of H_F at $(p, t) \in H_F^{-1}(0)$ is determined according to whether or not the map $dH_{F, (p, t)}(\cdot, 0) : T_p\Delta \rightarrow p^\perp$ preserves the orientations. Lemma 4 (i) implies $\tilde{K}_F^1(p^\lambda(r), g^\lambda(r), t^\lambda(r)) = 0$. Therefore considering the derivative of this equality with respect to r ,

$$d\tilde{K}_{F, (p^\lambda(r), g^\lambda(r), t^\lambda(r))}^1(dp_r^\lambda, dg_r^\lambda, 0) = -F(L^\lambda(r))dt_r^\lambda, \quad (3)$$

holds as observed in the proof of Lemma 4 in Appendix. Further as we observed above, Lemma 4 (ii) implies

$$d\tilde{K}_{F, (p^\lambda(r), g^\lambda(r), t^\lambda(r))}^1(v_l^\lambda(r), w_l^\lambda(r), 0) = \beta_l^\lambda(r), \quad l = 2, \dots, k. \quad (4)$$

Since $z^u(p^\lambda(r)) = \tilde{z}^1(\mathcal{L}(p^\lambda(r), g^\lambda(r)))$, we have $dz^u(dp_r^\lambda) = d\tilde{z}^1(d\mathcal{L}(dp_r^\lambda, dg_r^\lambda))$. Therefore, (3) implies that

$$dK_{F, (p^\lambda(r), g^\lambda(r), t^\lambda(r))}^1(dp_r^\lambda, dg_r^\lambda, 0) = -F(L^\lambda(r))dt_r^\lambda. \quad (5)$$

As seen in Claim 3 in Appendix, (4) implies the existence of $(v_l^{\lambda'}(r), w_l^{\lambda'}(r)) \in T_{(p^\lambda(r), t^\lambda(r))}V$ such that

$$dK_{F, (p^\lambda(r), t^\lambda(r), t^\lambda(r))}^1(v_l^{\lambda'}(r), w_l^{\lambda'}(r), 0) = \beta_l^\lambda(r), \quad l = 2, \dots, k. \quad (6)$$

Note that the vectors $-F(L^\lambda(r))dt_r^\lambda$ and $\beta_l^\lambda(r)$, $l = 2, \dots, k$, are on $L^\lambda(r)$. Furthermore, as in the proof of Lemma 3 in Appendix, there exists $M - k - 1$ independent vectors $(\bar{v}_l^\lambda(r), \bar{w}_l^\lambda(r)) \in T_{(p^\lambda(r), g^\lambda(r))}V$, $l = 1, \dots, M - k - 1$, such that $d\mathcal{L}_{(p^\lambda(r), g^\lambda(r))}(\bar{v}_l^\lambda(r), \bar{w}_l^\lambda(r)) = 0$; hence,

$$dK_{F, (p^\lambda(r), t^\lambda(r), t^\lambda(r))}^1(\bar{v}_l^\lambda(r), \bar{w}_l^\lambda(r), 0) = dz_{p^\lambda(r)}^u(\bar{v}_l^\lambda(r)), \quad l = 1, \dots, M - k - 1, \quad (7)$$

and the vectors $dz_{p^\lambda(r)}^1(\bar{v}_l^\lambda(r))$, $l = 1, \dots, M - k - 1$, are independent and not in $L^\lambda(r)$.¹⁵ From Lemma 2 (ii), the above (5)–(7) imply that at $p^\lambda(r) \in \Delta^g$,

$$\begin{aligned} dH_{F, (p^\lambda(r), t^\lambda(r))}(dp_r^\lambda, 0) &= -F(L^\lambda(r))dt_r^\lambda, \\ dH_{F, (p^\lambda(r), t^\lambda(r))}(v_l^{\lambda'}(r), 0) &= \beta_l^\lambda(r), \quad l = 2, \dots, k, \\ dH_{F, (p^\lambda(r), t^\lambda(r))}(\bar{v}_l^\lambda(r), 0) &= dz_{p^\lambda(r)}^u(\bar{v}_l^\lambda(r)), \quad l = 2, \dots, M - k - 1. \end{aligned}$$

In particular, at $(p^\lambda(0), 1)$, $dH_F(\cdot, 0)$ maps independent $M - 1$ vectors, dp_0^λ , $v_l^{\lambda'}(0)$, $l = 2, \dots, k$, and $\bar{v}_l^\lambda(0)$, $l = 1, \dots, M - k - 1$ to independent $M - 1$ vectors, $-F(L^\lambda(0))dt_0^\lambda$, $\beta_l^\lambda(0)$, $l = 2, \dots, k$, and $dz^u(\bar{v}_l^\lambda(0))$, $l = 1, \dots, M - k - 1$, respectively.

Therefore, if we take $-\beta_k^\lambda(r)$ in the place of $\beta_k^\lambda(r)$ leaving the others unchanged, and construct a new F , then with the new F , $dH_F(\cdot, 0)$ maps $v_k^{\lambda'}(0)$ to $-\beta_k^\lambda(0)$ at $(p^\lambda(0), 1)$; hence the orientation is reversed and the index of H_F at $(p^\lambda(0), 1)$ has a sign different from the previous one. This ends the construction of a smooth function $F : G_{++}^k(R^M) \rightarrow R^M$ satisfying Walras' law and (F1)–(F4).

7.2 Modification of the candidate function

Although we consider the function F as a candidate aggregate excess demand function of the constrained consumers, we cannot apply Proposition 2 to the function. First, F might not satisfy (BB) and (BC) of Proposition 2. Second, (F1) is slightly different from Proposition 3 (i) because (F1) does not exclude the case where $H_F(\cdot, \cdot) = 0$ has solutions other than the concerned paths.

We first modify F to a continuous function \bar{F} , which equals F in the neighborhood of $\bigcup_{\lambda, r} L^\lambda(r)$ and satisfies not only (W) but also (BB) and (BC). Let B and B' be open

¹⁵Recall the $M - k - 1$ -dimensional space $W_{(p, g)} \subset T_{(p, g)}V$ in the proof of Lemma 3 in Appendix. We take $(\bar{v}_l^\lambda(r), \bar{w}_l^\lambda(r)) \in W_{(p^\lambda(r), g^\lambda(r))}$.

subsets of $G_{++}^k(R^M)$ such that $\bigcup_{\lambda,r} L^\lambda(r) \subset B$ and $\bar{B} \subset B'$, and define $\bar{F} : G_{++}^k(R^M) \rightarrow R^M$ by

$$\bar{F}(L) = \eta(L)F(L) + (1 - \eta(L))f(L),$$

where η is a smooth function such that $\eta(L) = 1$ for $L \in B$, $\eta(L) = 0$ for $L \in G_{++}^k(R^M) \setminus B'$, and $f : G_{++}^k(R^M) \rightarrow R^M$ is an arbitrary smooth function satisfying (W),(BB), and (BC).

We next modify \bar{F} to z so that $K_z^1|V \times [0,1](p,g,t) = 0$ does not have solutions other than the paths $(p^\lambda(r), g^\lambda(r), t^\lambda(r))$. Since the projection of $(K_z^1|V \times [0,1])^{-1}(0)$ onto $\Delta \times [0,1]$ coincides with $\overline{H_z^{-1}(0)}$, if $(p^\lambda(r), g^\lambda(r), t^\lambda(r))$ are all solutions of $K_z^1|V \times [0,1](p,g,t) = 0$, then $\bigcup_{\lambda,r} (p^\lambda(r), t^\lambda(r)) = \overline{H_z^{-1}(0)}$ holds.

Let $D = \{(p^\lambda(r), g^\lambda(r), t^\lambda(r)) \in V \times [0,1] | r \in [0,1], \lambda = 1, \dots, \Lambda\}$ denote the set of the candidate paths and let $D' = \{(p,g,t) \in (V \times [0,1]) \setminus D | K_{\bar{F}}^1(p,g,t) = 0\}$ denote the set of the solutions of $K_{\bar{F}}^1|V \times [0,1](p,g,t) = 0$ other than these paths. Since we have constructed F such that $\hat{K}_F^1|V \times [0,1]$ is regular at any point on our paths, D and D' are disjoint closed sets.

Let $\tilde{D} = \bigcup_{\lambda,r} \mathcal{L}(p^\lambda(r), g^\lambda(r)) = \bigcup_{\lambda,r} L^\lambda(r) \subset G_{++}^k(R^M)$ denote the set of budget sets that appear on our paths and let $\tilde{D}' = \bigcup_{(p,g,t) \in D'} \mathcal{L}(p,g) \subset G_{++}^k(R^M)$ denote the set of budget sets that appear at other solutions of $K_{\bar{F}}^1|V \times [0,1](p,g,t) = 0$. These \tilde{D} and \tilde{D}' are closed subsets of $G_{++}^k(R^M)$ as projections of the closed subsets D and D' .

We observe that \tilde{D} and \tilde{D}' are disjoint. To the contrary, suppose $\bar{L} \in \tilde{D} \cap \tilde{D}'$. Then there exists $(\bar{p}, \bar{g}, \bar{t}) \in D'$ such that $\bar{L} = \mathcal{L}(\bar{p}, \bar{g})$ and

$$z^u(\bar{p}) + \bar{t}\bar{F}(\bar{L}) = 0 \tag{8}$$

on the one hand, and $\bar{L} = \mathcal{L}(p^{\bar{\lambda}}(\bar{r}), g^{\bar{\lambda}}(\bar{r}))$ for some $\bar{\lambda}$ and \bar{r} and

$$z^u(p^{\bar{\lambda}}(\bar{r})) + \bar{t}^{\bar{\lambda}}(\bar{r})\bar{F}(\bar{L}) = 0 \tag{9}$$

on the other hand. Although (8) and (9) imply that $z^u(\bar{p})$ is parallel to $z^u(p^{\bar{\lambda}}(\bar{r}))$, this holds true only if $\bar{p} = p^{\bar{\lambda}}(\bar{r})$ and $z^u(\bar{p}) = z^u(p^{\bar{\lambda}}(\bar{r}))$ because of the revealed axiom of the first consumer's strictly convex preference. Note that $\bar{p} = p^{\bar{\lambda}}(\bar{r})$ and $\mathcal{L}(\bar{p}, \bar{g}) = \mathcal{L}(p^{\bar{\lambda}}(\bar{r}), g^{\bar{\lambda}}(\bar{r}))$ imply $\bar{g} = g^{\bar{\lambda}}(\bar{r})$, as shown in Claim 1 in Appendix. Then, $\bar{t} \neq \bar{t}^{\bar{\lambda}}(\bar{r})$ should hold for D and D' to be disjoint sets. However, (8) and (9) with $\bar{t} \neq \bar{t}^{\bar{\lambda}}(\bar{r})$ implies $z^u(\bar{p}) = z^u(p^{\bar{\lambda}}(\bar{r})) = 0$ and $\bar{F}(\bar{L}) = 0$, and hence, $z^u(\bar{p}) + \bar{t}\bar{F}(\bar{L}) = 0$ holds for any t . This contradicts to that D and D' are disjoint. Thus, \tilde{D} and \tilde{D}' are disjoint.

Let t^* denotes the minimum of t such that $(p,g,t) \in D'$. Note that t^* is positive because $(p^1(0), g^1(0)) = (p^u, \langle A(p^u) \rangle)$ is the unique solution of $K_{\bar{F}}^1|V \times [0,1](p,g,0) = 0$.

Since \tilde{D} and \tilde{D}' are disjoint closed sets in $G_{++}^k(R^M)$, we let C be an open set such that it includes \tilde{D}' in its interior and its closure and \tilde{D} are disjoint. We define a smooth function $\phi : G_{++}^k(R^M) \rightarrow (0, 1]$ such that $\phi(L) = 1$ for $L \notin C$ and $\phi(L) < t^*$ for $L \in \tilde{D}'$, and we define $z : G_{++}^k(R^M) \rightarrow R^M$ as

$$z(L) = \phi(L)\bar{F}(L).$$

For $(p, g, t) \in V \times [0, 1]$ such that $\mathcal{L}(p, g) \notin C$, $K_z^1(p, g, t) = K_{\bar{F}}^1(p, g, t)$. Therefore, the paths $(p^\lambda(r), g^\lambda(r), t^\lambda(r))$ are all solutions of $K_z^1|V \times [0, 1](p, g, t) = 0$ in $\{(p, g, t) \in V \times [0, 1] | \mathcal{L}(p, g) \notin C\}$.

For $(p, g, t) \in V \times [0, 1]$ such that $\mathcal{L}(p, g) \in C$, if $K_z^1(p, g, t) = z^u(p) + t\phi(\mathcal{L}(p, g))F(\mathcal{L}(p, g)) = 0$, then $(p, g, t\phi(\mathcal{L}(p, g))) \in D'$. However, $t\phi(\mathcal{L}(p, g)) < t^*$ from the definition of ϕ and $t \in [0, 1]$. Therefore, $(p, g, t\phi(\mathcal{L}(p, g))) \in D'$ contradicts the definition of t^* . Thus, $K_z^1|V \times [0, 1](p, g, t) = 0$ has no solution in $\{p, g, t \in V \times [0, 1] | \mathcal{L}(p, g) \in C\}$.

Thus, we have proved that the paths $(p^\lambda(r), g^\lambda(r), t^\lambda(r))$ are all solutions of $K_z^1|V \times [0, 1](p, g, t) = 0$. It is clear from the construction that $z : G_{++}^k(R^M) \rightarrow R^M$ is a smooth function that equals F in the neighborhood of $\bigcup_{\lambda, r} L^\lambda(r)$ and satisfies (W), (BB), and (BC).

7.3 Construction of constrained consumers

We apply proposition 2 to z and obtain a set of constrained consumers that establishes Proposition 3. According to Momi (2010), if $(p, g) \in V$, then $\mathcal{L}(p, g) \in G_{++}$, and if $L \in G_{++}$, then there exists $(p, g) \in V$ such that $L = \mathcal{L}(p, g)$. That is, $\mathcal{L}(V) = G_{++}$.¹⁶

We select a sufficiently small μ and, by applying Proposition 2 to the function z , we obtain a set of a finite number of consumers $\{\tilde{z}^i, \omega^i\}_{i \geq 2}$ whose aggregate excess demand function z^* satisfies $z^*|G_\mu = z|G_\mu$ and $\sum_l z_l^*(L) > 0$ for $L \in G_{++} \setminus G_\mu$. Without loss of generality, we can assume that μ is sufficiently small so that $\bigcup_{\lambda, r} L^\lambda(r) \subset G_\mu$ and $\sum_{l=1}^M \tilde{z}_l^1(L) > 0$ for $L \in G_{++} \setminus G_\mu$.

Since $\mathcal{L}(V) = G_{++}$, decompose V into $V' = \{(p, g) \in V | \mathcal{L}(p, g) \in G_\mu\}$ and $V'' = \{(p, g) \in V | \mathcal{L}(p, g) \in G_{++} \setminus G_\mu\}$.

¹⁶If $p \in \Delta^g$, then g satisfying $(p, g) \in V$ is uniquely determined as $g = g(p) = \langle A(p) \rangle$; hence, the statement is trivial. Let $\bar{p} \in \Delta^b$. If there exists a sequence of good prices converging to bad price $p \rightarrow \bar{p}$ whose budget set converges to \bar{L} , then by defining $g(\bar{p}) = \lim_{p \rightarrow \bar{p}} g(p)$, we have $(\bar{p}, g(\bar{p})) \in V$. On the other hand, if $(\bar{p}, \bar{g}) \in V$, then $\mathcal{L}(\bar{p}, \bar{g})^\perp$ is an $S - J + 1$ -dimensional space that includes \bar{p} and is included in $L(\bar{p})^\perp$. Thus, by Result 2 in Momi (2010), there exists a sequence of good prices converging to \bar{p} whose budget set converges to $\mathcal{L}(\bar{p}, \bar{g})$, that is, $\mathcal{L}(\bar{p}, \bar{g}) \in G_{++}$.

If $L \in G_{++} \setminus G_\mu$, then $\sum_{l=1}^M \tilde{z}_l^1(L) > 0$ and $\sum_{l=1}^M z_l^*(L) > 0$; hence, with any $t \in [0, 1]$, $\tilde{z}^1(L) + tz^*(L) = 0$ has no solution in $G_{++} \setminus G_\epsilon$. This implies that $\tilde{z}^1(\mathcal{L}(p, g)) + tz^*(\mathcal{L}(p, g)) = 0$ has no solution in $V'' \times [0, 1]$, and hence, $z^u(p) + tz^*(\mathcal{L}(p, g)) = 0$ has no solution in $V'' \times [0, 1]$. On the other hand, since $z^*|_{G_\mu} = z|_{G_\mu}$ and $\bigcup_{\lambda, r} L^\lambda(r) \subset G_\mu$, $K_{z^*}^1(p, g, t) = K_z^1(p, g, t)$ for $(p, g, t) \in V' \times [0, 1]$, and the paths $(p^\lambda(r), g^\lambda(r), t^\lambda(r))$ are all solutions of $K_{z^*}|V \times [0, 1](p, g, t) = 0$ in $V' \times [0, 1]$. These establish that the set of consumers $\{\tilde{z}^i, w^i\}_{i \geq 2}$ satisfies Proposition 3 (i).

Since $K_{z^*}^1|V \times [0, 1](p, g, t) = K_z^1|V \times [0, 1](p, g, t) = K_F^1|V \times [0, 1](p, g, t)$ for (p, g, t) in the neighborhood of the paths $(p^\lambda(r), g^\lambda(r), t^\lambda(r))$ and F is constructed to satisfy (F2)–(F4), the set of consumers $\{\tilde{z}^i, w^i\}_{i \geq 2}$ satisfies (ii)–(iv) of Proposition 3. This ends the proof of Proposition 3.

A Appendix

A.1 Proof of Lemma 1

We show an example of paths satisfying the conditions.

We construct the paths such that the first path has no bad points whereas the other paths pass through one bad point at $r = \frac{1}{2}$. Then, the index at the end point of the first path on $\Delta \times \{1\}$ is +1, and both the end points of the other paths have the same index (+1 or -1) under the assumption that the index change occurs at the bad point at $r = \frac{1}{2}$. Therefore, we determine the number of paths Λ to be $|\alpha - 1|/2$, which is the least number to satisfy (P1).

First, we choose $(\tilde{p}^\lambda(r), t^\lambda(r))$, $\lambda = 1, \dots, \Lambda$, satisfying all the conditions except (h7).

We define $\tilde{p}^1(r) = (p_0^u, \dots, p_{S-1}^u, p_S^u + rb)$, where $p^u = (p_0^u, \dots, p_S^u)$ is the supporting price of the indifference surface of the first consumer's preference at her initial endowment w^1 and $b = (b_1, \dots, b_N)$ is a sufficiently small vector that is not parallel to p_S^u . If b is sufficiently small, then $\tilde{p}^1(r)$ is in the neighborhood of $p^u \in \Delta^g$ for any $r \in [0, 1]$; hence, the first path passes through no bad points.

For each path $\lambda = 2, \dots, \Lambda$, we define $\tilde{p}^\lambda(r)$ as follows. For state $s = 0$, we select $\bar{p}_0^2 = (\bar{p}_{01}^2, \dots, \bar{p}_{0N}^2), \dots, \bar{p}_0^\Lambda = (\bar{p}_{01}^\Lambda, \dots, \bar{p}_{0N}^\Lambda)$, so that each of them is parallel to neither the others nor $p_0^u = (p_{01}^u, \dots, p_{0N}^u)$, and we define $\tilde{p}_0^\lambda(r) = \bar{p}_0^\lambda$, $\lambda = 2, \dots, \Lambda$. For other states $s = 1, \dots, S$, we select a bad price $\bar{p} = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_S) \in \Delta^b$ and define $\tilde{p}_s^\lambda(r) = \bar{p}_s$ for $s = 1, \dots, S - 1$, and $\tilde{p}_S^\lambda(r) = \bar{p}_S + (r - \frac{1}{2})a$ for $s = S$, where $a = (a_1, \dots, a_N) \in R^N$

is a vector such that the determinant of the matrix $D(a)$ defined as

$$D(a) = \begin{bmatrix} \bar{p}_{S-J+1}A_{S-J+1}^1 & \cdots & \bar{p}_{S-J+1}A_{S-J+1}^J \\ \vdots & & \vdots \\ \bar{p}_{S-1}A_{S-1}^1 & \cdots & \bar{p}_{S-1}A_{S-1}^J \\ aA_S^1 & \cdots & aA_S^J \end{bmatrix}$$

is not zero: $\det D(a) \neq 0$.

Let $id \in \Sigma$ denote the identical permutation of $\{1, \dots, S\}$. Then, $Q_{id}(p)$ denotes a matrix consisting of the last J rows of $A(p)$. The determinant of this matrix for the paths $\lambda = 2, \dots, \Lambda$, is then calculated as

$$\det[Q_{id}(\tilde{p}^\lambda(r))] = (r - \frac{1}{2})\det D(a),$$

which is of full rank for any $r \in [0, 1]$ except $r = \frac{1}{2}$ and changes its signs at $r = \frac{1}{2}$.

Finally, we define $t^\lambda(r)$ as arbitrary smooth functions whereby $t^1(0) = 0$ and $t^1(1) = 1$ for $\lambda = 1$; $t^\lambda(0) = t^\lambda(1) = 1$ for $\lambda = 2, \dots, \Lambda$; and $0 < t^\lambda(r) < 1$ for any $r \in (0, 1)$ and any λ .

It can be seen that the paths $(\tilde{p}^\lambda(r), t^\lambda(r))$, $\lambda = 1, \dots, \Lambda$, satisfy the conditions except (h7). It is clear that (h1)–(h6) are satisfied. For (h8) and (h9), we apply the result given by Bottazzi and Hens (1996) [Proposition 1 and its proof. Let p and p' be price vectors in Δ . Then, $L(p) = L(p')$ if and only if $p' \in L(p)^\perp$; further, $p' \in L(p)^\perp$ if and only if $p' = tp + (0, \beta_1 p_1, \dots, \beta_S p_S)$ with parameter t and a vector $(\beta_1, \dots, \beta_S) \in R^S$ such that $[\beta_1, \dots, \beta_S]A(p) = 0$]. Note that if $L(p) = L(p')$, then the spot price of the former p_s should be parallel to that of the latter p'_s at each states $s = 0, \dots, S$.

Returning to (h8), for a path λ' and another λ'' , $\tilde{p}_0^{\lambda'}(r) = \bar{p}_0^{\lambda'}$ is not parallel to $\tilde{p}_0^{\lambda''}(r) = \bar{p}_0^{\lambda''}$. Therefore, $L(\tilde{p}^{\lambda'}(r')) \neq L(\tilde{p}^{\lambda''}(r''))$ if $\lambda' \neq \lambda''$ for any r' and r'' . For each $\lambda = 2, \dots, \Lambda$, $\tilde{p}_S^\lambda(r) = \bar{p}_S + (r - \frac{1}{2})a$ and \bar{p}_S and a are independent; therefore, $\tilde{p}_S^\lambda(r')$ is not parallel to $\tilde{p}_S^\lambda(r'')$. For $\lambda = 1$, $\tilde{p}_S^1(r')$ is not parallel to $\tilde{p}_S^1(r'')$ because p_S^u and b are not parallel. Therefore, $L(\tilde{p}^\lambda(r')) \neq L(\tilde{p}^\lambda(r''))$ if $r' \neq r''$. These establish (h8).

As for (h9), observe that $d\tilde{p}_r^1 = (0, \dots, 0, b)$, $d\tilde{p}_r^\lambda = (0, \dots, 0, a)$, $\lambda = 2, \dots, \Lambda$, b is not parallel to $\tilde{p}_S^1(r)$ and a is not parallel to $\tilde{p}_S^\lambda(r)$, $\lambda = 2, \dots, \Lambda$. Thus $d\tilde{p}_r^\lambda \notin L(\tilde{p}^\lambda(r))^\perp$. In other words, $d\tilde{L}^\lambda \neq 0$, where $\tilde{L}^\lambda(r)$ is defined as $\tilde{L}^\lambda(r) = L^\lambda(\tilde{p}^\lambda(r))$ for r such that $p^\lambda(r)$ is a good price and $\tilde{L}^\lambda(r') = \lim_{r \rightarrow r'} L(\tilde{p}^\lambda(r))$ for r' such that $p^\lambda(r')$ is a bad price.

As for (P2), each path, $\lambda = 2, \dots, \Lambda$, passes through one bad point at $r = \frac{1}{2}$ and $\det[Q_{id}(\tilde{p}^\lambda(r))]$ changes its sign at $r = \frac{1}{2}$. To complete (P2), it is necessary to show that $\langle A(\tilde{p}(r)) \rangle \in W_{id}$ for r around $r = \frac{1}{2}$. Thus it should be shown that $Y \in R^{(S-J)J}$ satisfying

$[I|Y]A(\tilde{p}^\lambda(r)) = 0$ is well defined for r around $r = \frac{1}{2}$. From straightforward calculations using Cramer's rule, we obtain

$$Y_{kl} = -\frac{\det Q_{id}^{kl}(\tilde{p}^\lambda(r))}{\det Q_{id}(\tilde{p}^\lambda(r))}, \quad k = 1, \dots, S - J, \quad l = 1, \dots, J,$$

where $Q_{id}^{kl}(\tilde{p}^\lambda(r))$ is the matrix $Q_{id}(\tilde{p}^\lambda(r))$ whose l -th row, $[\tilde{p}_{S-J+l}^\lambda(r)A_{S-J+l}^1, \dots, \tilde{p}_{S-J+l}^\lambda(r)A_{S-J+l}^J]$, is replaced with the k -th row of $A(\tilde{p}^\lambda(r))$, $[\tilde{p}_k^\lambda(r)A_k^1, \dots, \tilde{p}_k^\lambda(r)A_k^J]$. Then,

$$Y_{kl} = \begin{cases} -\frac{\det D^{kl}(a)(r-\frac{1}{2})}{\det D(a)(r-\frac{1}{2})} = -\frac{\det D^{kl}(a)}{\det D(a)}, & \text{for } l \neq J \\ 0, & \text{for } l = J \end{cases},$$

where $D^{kl}(a)$ is the matrix $D(a)$ whose l -th row is replaced with $[\bar{p}_k A_k^1, \dots, \bar{p}_k A_k^J]$. Thus, Y satisfying $[I|Y]A(\tilde{p}(r))$ is well defined for any r and any $\lambda = 2, \dots, \Lambda$. This supports our choice of $(\tilde{p}^\lambda(r), t^\lambda(r))$.

We next transform $(\tilde{p}^\lambda(r), t^\lambda(r))$ to $(p^\lambda(r), t^\lambda(r))$ satisfying all the requirements including (h7). Let us define $p^\lambda(r)$ as the supporting price of the first consumer's indifference surface at $\tilde{z}^1(\tilde{L}^\lambda(r)) + w^1$

It is clear that (h1)–(h4) and (h7) are satisfied with $(p^\lambda(r), t^\lambda(r))$, $\lambda = 1, \dots, \Lambda$. For the other conditions, note that $p^\lambda(r)$ is perpendicular to $L(\tilde{p}^\lambda(r))$ by definition. Then, $L(p^\lambda(r)) = L(\tilde{p}^\lambda(r))$ from the proposition by Bottazzi and Hens (1996). Therefore, $p^\lambda(r)$ is a good (bad) price when $\tilde{p}^\lambda(r)$ is a good (bad) price. This establishes (h5) and (h6).

If $L(p^{\lambda'}(r')) = L(p^{\lambda''}(r''))$, then $L(\tilde{p}^{\lambda'}(r')) = L(\tilde{p}^{\lambda''}(r''))$; hence, $(\lambda', r') = (\lambda'', r'')$ because $\tilde{p}^\lambda(r)$ satisfies (h8). Thus, (h8) is satisfied.

We observed (h9) for $\tilde{p}^\lambda(r)$. $L(p^\lambda(r)) = L(\tilde{p}^\lambda(r))$ and $d\tilde{L}_r^\lambda \neq 0$ implies $dL_r^\lambda \neq 0$, that is, $p^\lambda(r)$ satisfies (h9).

As for (P2), $p^\lambda(r)$ is a bad price at $r = \frac{1}{2}$ for $\lambda = 2, \dots, \Lambda$. Since $p_s^\lambda(r)$ is parallel to $\tilde{p}_s^\lambda(r)$, $s = 0, \dots, S$, $p_s^\lambda(r) = \alpha_s^\lambda(r)(\tilde{p}_s^\lambda(r))$ with some non-zero parameter $\alpha_s^\lambda(r)$ at each r . Using these parameters, we can calculate that

$$\det Q_{id}(p^\lambda(r)) = (\prod_{s=S-J+1}^S \alpha_s^\lambda(r))(r - \frac{1}{2})\det D(a),$$

which changes its sign at $r = \frac{1}{2}$. By the same calculations in the case of $\tilde{p}^\lambda(r)$, Y satisfying $[I|Y]A(p^\lambda(r)) = 0$ for r around $\frac{1}{2}$ is calculated as $Y_{kl} = \det Q_{id}^{kl}(p^\lambda(r))/\det Q_{id}(p^\lambda(r))$, $k = 1, \dots, S - J$, $l = 1, \dots, J$, that is,

$$Y_{kl} = \begin{cases} -\frac{\alpha_k^\lambda(r)\det D^{kl}(a)}{\alpha_l^\lambda(r)\det D(a)}, & \text{for } l \neq J \\ 0, & \text{for } l = J \end{cases},$$

which is well defined for any r . Thus, (P2) is satisfied.

A.2 Proof of Lemma 2

At any $(p, t) \in \Delta^g \times [0, 1]$, $H_F(p, t) = z^u(p) + F(L(p)) \in p^\perp$ because $z^u(p) \in p^\perp$ as the result of utility maximization and $F(L(p)) \in L(p) \subset p^\perp$ as F is assumed to satisfy $F(L) \in L$. Hence $dH_{(p,t)}$ maps to p^\perp at $(p, t) \in H^{-1}(0)$. Therefore \hat{H}_F at $(p, t) \in H^{-1}(0)$ is regular when $dH_{(p,t)}$ is surjective onto p^\perp .

Similarly, $K_{\sigma F}^1(p, Y, t) \in p^\perp$; hence $dK_{\sigma F, (p, \varphi_\sigma(g), t)}$ maps to $p^\perp \times TR^{(S-J)J}$ at $(p, \varphi_\sigma(g), t) \in K_{\sigma F}^{-1}(0)$ with $g \in W_\sigma$. Therefore $\hat{K}_{\sigma F}$ is regular at $(p, \varphi_\sigma(g), t) \in K_{\sigma F}^{-1}(0)$ when $dK_{\sigma F, (p, \varphi_\sigma(g), t)}$ is surjective onto $p^\perp \times TR^{(S-J)J}$.

(i) From the definitions, $K_F^1(p, g, t) = K_{\sigma F}^1(p, \varphi_\sigma(g), t)$ for σ satisfying $g \in W_\sigma$ and $K_{\sigma F}^2(p, \varphi_\sigma(g)) = 0$ for $(p, g) \in V$. Therefore, if $K_F^1(p, g, t) = 0$ for $(p, g) \in V$, then $K_{\sigma F}^2(p, \varphi_\sigma(g), t) = 0$ for σ satisfying $g \in W_\sigma$.

Let us define $V_\sigma = \{(p, \varphi_\sigma(g)) \in \Delta \times R^{(S-J)J} \mid (p, g) \in V, g \in W_\sigma\}$. It is clear from $K_F^1(p, g, t) = K_{\sigma F}^1(p, \varphi_\sigma(g), t)$ that $dK_{F, (p, g, t)}^1(T_{p, g, t}(V \times [0, 1])) = dK_{\sigma F, (p, \varphi_\sigma(g), t)}^1(T_{(p, \varphi_\sigma(g), t)}(V_\sigma \times [0, 1]))$. It is also clear that $dK_{\sigma F, (p, \varphi_\sigma(g))}^2(T_{(p, \varphi_\sigma(g))}V_\sigma) = 0$. Moreover, from $K_{\sigma F, (p, \varphi_\sigma(g))}^2 = [I \mid \varphi_\sigma(g)] \Pi_\sigma A(p)$, we have

$$\left[\frac{\partial K_{\sigma F}^2}{\partial p_{\sigma(1)}}, \dots, \frac{\partial K_{\sigma F}^2}{\partial p_{\sigma(S-J)}} \right] = \begin{bmatrix} A_{\sigma(1)}^T & & \\ & \ddots & \\ & & A_{\sigma(S-J)}^T \end{bmatrix},$$

where the $J \times N$ matrices

$$A_{\sigma(s)}^T = \begin{bmatrix} A_{\sigma(s)1}^1 & \cdots & A_{\sigma(s)N}^1 \\ \vdots & & \vdots \\ A_{\sigma(s)1}^J & \cdots & A_{\sigma(s)N}^J \end{bmatrix}, \quad s = 1, \dots, S - J,$$

line diagonally and other elements are zero's. Therefore, if (A3) is satisfied, $dK_{\sigma F}^2$ is always surjective onto $TR^{(S-J)J}$. These prove (i).

(ii) Let $p \in \Delta^g$. Then, $(p, g) \in V$ implies that $g = g(p) \equiv \langle A(p) \rangle$, and hence, $\mathcal{L}(p, g) = L(p)$. Therefore, $H_F(p, t) = z^u(p) + tF(L(p)) = z^u(p) + tF(\mathcal{L}(p, g)) = K_F^1(p, g, t)$. Similarly, $(v, w) \in T_{(p,g)}V$ implies $w = dg_p(v)$, where dg_p denotes the derivative of $p \rightarrow g(p) \equiv \langle A(p) \rangle$ at p ; hence, $d\mathcal{L}_{(p,g)}(v, w) = dL_p(v)$ for $(v, w) \in T_{(p,g)}V$. Therefore, $d(K_F^1|V \times [0, 1])_{(p,g,t)}(v, w, a) = dz_p^1(v) + tdF_{\mathcal{L}(p,g)}d\mathcal{L}_{(p,g)}(v, dg_p(v)) + F(\mathcal{L}(p, g))a = dz_p^1(v) + tdF_{L(p)}dL_p(v) + F(L(p))a = dH_{F, (p,t)}(v, a)$. These prove (ii).

A.3 Proof of Lemma 3

We first summarize preliminary results which are essentially same as the proposition in Bottazzi and Hens (1996) we mentioned in the proof of Lemma 1.

Claim 1. (i) For any $p \in \Delta$ and $g \in G^J(R^S)$, $\pi \in \mathcal{L}(p, g)^\perp$ if and only if $\pi = tp + (0, \beta_1 p_1, \dots, \beta_S p_S)$ with the parameters $t \in R$ and $(\beta_1, \dots, \beta_S) \in g^\perp$, where g^\perp denotes the perpendicular space of g .

(ii) For any $p \in \Delta$ and $g \in G^J(R^S)$, $\mathcal{L}(p, g) = \mathcal{L}(p, g')$ if and only if $g = g'$.

(iii) For any $p, p' \in \Delta$ and $g, g' \in G^J(R^S)$, if $\mathcal{L}(p, g) = \mathcal{L}(p', g')$, then $p' \in \mathcal{L}(p, g)^\perp$.

(iv) For any price vector $p' \in \mathcal{L}(p, g)^\perp \cap \Delta$, there exists a unique g' such that $\mathcal{L}(p', g') = \mathcal{L}(p, g)$. If $\langle A(p) \rangle \subset g$, then $\langle A(p') \rangle \subset g'$.

(v) For any $v \in \mathcal{L}(p, g)^\perp \cap T_p \Delta$, there exists a unique $w \in T_g G^J(R^S)$ such that $d\mathcal{L}_{(p, g)}(v, w) = 0$. If $(p, g) \in V$, then $(v, w) \in T_{(p, g)} V$.

(vi) If $v, v' \in \mathcal{L}(p, g)$ are independent vectors, then $d\mathcal{L}_{(p, g)}(v, w)$ and $d\mathcal{L}_{(p, g)}(v', w')$ are independent vectors for any $w, w' \in T_g G^J(R^S)$.

Proof. (i) This is a restatement of Proposition 1 in Bottazzi and Hens (1996). We define $\Pi(p, g) = \{\pi \in R^M \mid \pi = tp + (0, \beta_1 p_1, \dots, \beta_S p_S), t \in R, (\beta_1, \dots, \beta_S) \in g^\perp\}$ and prove that $\Pi(p, g) = \mathcal{L}(p, g)^\perp$.

Recall the definition of the budget set $\mathcal{L}(p, g) = \{z \in R^M \mid pz = 0, p_1 \square z_1 \in g\}$. The first condition $pz = 0$ implies that p is perpendicular to $z \in \mathcal{L}(p, g)$. The second condition $p_1 \square z_1 \in g$ implies that $\beta_1 p_1 z_1 + \dots + \beta_S p_S z_S = 0$ for any $(\beta_1, \dots, \beta_S) \in g^\perp$, that is, it implies that $(0, \beta_1 p_1, \dots, \beta_S p_S)$ is perpendicular to $z \in \mathcal{L}(p, g)$. Therefore $\pi \in \Pi(p, g)$, which is a linear combination of p and $(0, \beta_1 p_1, \dots, \beta_S p_S)$, is perpendicular to $\mathcal{L}(p, g)$; hence $\Pi(p, g) \subset \mathcal{L}(p, g)^\perp$. On the other hand, recall that $\mathcal{L}(p, g)^\perp$ is an $S - J + 1$ -dimensional linear space and note that $\Pi(p, g)$ is also an $S - J + 1$ -dimensional linear space because g^\perp is an $S - J$ -dimensional space in R^S . These imply $\Pi(p, g) = \mathcal{L}(p, g)^\perp$.

(ii) $\mathcal{L}(p, g) = \mathcal{L}(p, g')$ implies $\mathcal{L}(p, g)^\perp = \mathcal{L}(p, g')^\perp$. Therefore, if $\mathcal{L}(p, g) = \mathcal{L}(p, g')$, then, because of (i), for any $(\beta_1, \dots, \beta_S) \in g^\perp$ there must exist $(\beta'_1, \dots, \beta'_S) \in g'^\perp$ such that $(\beta_1 p_1, \dots, \beta_S p_S) = (\beta'_1 p_1, \dots, \beta'_S p_S)$. This is satisfied only when $(\beta_1, \dots, \beta_S) = (\beta'_1, \dots, \beta'_S)$, that is, $g = g'$.

(iii) If $\mathcal{L}(p, g) = \mathcal{L}(p', g')$, then $\mathcal{L}(p, g)^\perp = \mathcal{L}(p', g')^\perp$. It is clear that $p' \in \mathcal{L}(p', g')^\perp$. Thus, $p' \in \mathcal{L}(p, g)^\perp$.

(iv) When $p' \in \mathcal{L}(p, g)^\perp \cap \Delta$, p' should be written, because of (i), as $p' = \bar{t}p + (0, \bar{\beta}_1 p_1, \dots, \bar{\beta}_S p_S)$, where $(\bar{\beta}_1, \dots, \bar{\beta}_S) \in g^\perp$, and \bar{t} should not be zero for p' to be a price vector in Δ . For any $(\beta_1, \dots, \beta_S) \in g^\perp$, we define $\beta'_1, \dots, \beta'_S$ as

$$(\beta'_1 p'_1, \dots, \beta'_S p'_S) = (\beta'_1 (\bar{t} + \bar{\beta}_1) p_1, \dots, \beta'_S (\bar{t} + \bar{\beta}_S) p_S) = (\beta_1 p_1, \dots, \beta_S p_S), \quad (10)$$

that is, we define $\beta'_s = \frac{\beta_s}{\bar{t} + \bar{\beta}_s}$, $s = 1, \dots, S$. We then define g' such that $(g')^\perp$ is the set of

$(\beta'_1, \dots, \beta'_S) \in R^S$:

$$(g')^\perp = \left\{ (\beta'_1, \dots, \beta'_S) \in R^S \mid \beta'_s = \frac{\beta_s}{\bar{t} + \bar{\beta}_s}, s = 1, \dots, S, (\beta_1, \dots, \beta_S) \in g^\perp \right\}.$$

Note that since g^\perp is an $S - J$ -dimensional linear space, $(g')^\perp$ as defined above is also $S - J$ -dimensional; hence, g' is a J -dimensional linear space in R^S .

To prove $\mathcal{L}(p', g') = \mathcal{L}(p, g)$, it is sufficient to show $\mathcal{L}(p, g)^\perp \subset \mathcal{L}(p', g')^\perp$ because both are the linear spaces of the same dimension. First, any vector $(0, \beta_1 p_1, \dots, \beta_S p_S) \in \mathcal{L}(p, g)^\perp$ with any $(\beta_1, \dots, \beta_S) \in g^\perp$ is in $\mathcal{L}(p', g')^\perp$ because $(\beta'_1, \dots, \beta'_S) \in g'^\perp$ satisfies (10) and $(0, \beta_1 p_1, \dots, \beta_S p_S) = (0, \beta'_1 p'_1, \dots, \beta'_S p'_S) \in \mathcal{L}(p', g')^\perp$. Second, $p' = \bar{t}p + (0, \bar{\beta}_1 p_1, \dots, \bar{\beta}_S p_S)$ is clearly in $\mathcal{L}(p', g')^\perp$ and $(0, \bar{\beta}_1 p_1, \dots, \bar{\beta}_S p_S)$ is also in $\mathcal{L}(p', g')^\perp$, as shown above. Hence, $p \in \mathcal{L}(p', g')^\perp$. These prove $\mathcal{L}(p, g)^\perp \subset \mathcal{L}(p', g')^\perp$; hence $\mathcal{L}(p', g') = \mathcal{L}(p, g)$. Uniqueness of g' follows (ii).

To prove the last statement, it should be shown that $(p_1 A_1^j, \dots, p_S A_S^j) \in g$ implies $(p'_1 A_1^j, \dots, p'_S A_S^j) \in g'$ for any $j = 1, \dots, J$. If $(p_1 A_1^j, \dots, p_S A_S^j) \in g$, then, by any $(\beta_1, \dots, \beta_S) \in g^\perp$ and the corresponding $(\beta'_1, \dots, \beta'_S) \in g'^\perp$ satisfying (10), we have $\beta_1 p_1 A_1^j + \dots + \beta_S p_S A_S^j = \beta'_1 p'_1 A_1^j + \dots + \beta'_S p'_S A_S^j = 0$. This implies $(p'_1 A_1^j, \dots, p'_S A_S^j) \in g'$.

(v) This is a restatement of (iv).

(vi) Suppose $d\mathcal{L}_{(p,g)}(v, w) = td\mathcal{L}_{(p,g)}(v', w')$ with a scalar t . Then, $d\mathcal{L}_{(p,g)}(v - tv', w - tw') = 0$, which happens only if $v - tv' \in \mathcal{L}(p, g)^\perp$ as (iii) shows. However, that v and v' are independent vectors in $\mathcal{L}(p, g)$ implies that $v - tv'$ is a non-zero vector in $\mathcal{L}(p, g)$, not in $\mathcal{L}(p, g)^\perp$. ■

To consider the difference between $\tilde{K}_F^1(p, g, t) = \tilde{z}^1(\mathcal{L}(p, g)) + tF(\mathcal{L}(p, g))$ and $K_F^1(p, g, t) = z^u(p) + tF(\mathcal{L}(p, g))$, we use the next result.

Claim 2. For any non-zero $v \in T_p \Delta$ such that $v^T z^u(p) = 0$, $v^T dz_p^u(v) < 0$.

Proof. See Geanakoplos and Polemarchakis (1980) [Proposition A and Corollary A, p. 319].

Claim 3. Let $(p, g, t) \in V \times [0, 1]$ satisfy $\tilde{K}_F^1(p, g, t) = K_F^1(p, g, t) = 0$. For any $(v', w', a) \in T_{(p,g,t)}(V \times [0, 1])$, there exists $(v'', w'') \in T_{(p,g)}V$ such that $d\tilde{K}_{F,(p,g,t)}^1(v', w', a) = dK_{F,(p,g,t)}^1(v'', w'', a)$.

Proof. It is sufficient to prove that for any $(p', g') \in V$ in the neighborhood of (p, g) , there exists $(p'', g'') \in V$ in the neighborhood of (p, g) such that $\tilde{K}_F^1(p', g', t) = K_F^1(p'', g'', t)$.

Suppose that $(p', g') \in V$ is given. Let p'' be the supporting price of the indifference surface of the first consumer's preference at $\tilde{z}^1(\mathcal{L}(p', g'))$ so that $\tilde{z}^1(\mathcal{L}(p', g')) = z^u(p'')$.

Since p'' is perpendicular to $\mathcal{L}(p', g')$, there is a unique g'' such that $(p'', g'') \in V$ and $\mathcal{L}(p'', g'') = \mathcal{L}(p', g')$ as seen in Claim 1 (iv); hence, $F(\mathcal{L}(p', g')) = F(\mathcal{L}(p'', g''))$. Thus, $\tilde{K}_F^1(p', g', t) = K_F^1(p'', g'', t)$.

When $\tilde{K}_F^1(p, g, t) = K_F^1(p, g, t) = 0$, $z^u(p) = \tilde{z}^1(\mathcal{L}(p, g))$. That is, p is the supporting price of the indifference surface of the first consumer's preference at $\tilde{z}^1(\mathcal{L}(p, g))$. Therefore, if (p', g') is in the neighborhood of (p, g) , then $\mathcal{L}(p', g')$ is in the neighborhood of $\mathcal{L}(p, g)$, and p'' is in the neighborhood of p . Moreover, since $\mathcal{L}(p', g') = \mathcal{L}(p'', g'')$ is in the neighborhood of $\mathcal{L}(p, g)$, (p'', g'') should be in the neighborhood of (p, g) . ■

Proof of Lemma 3. We suppose that $d(\tilde{K}_F^1|V \times [0, 1])$ is surjective onto $\mathcal{L}(p, g)$ at (p, g, t) satisfying $\tilde{K}_F^1(p, g, t) = K_F^1(p, g, t) = 0$. From Claim 3, $\mathcal{L}(p, g) = d\tilde{K}_{F,(p,g,t)}^1(T_{(p,g,t)}(V \times [0, 1])) \subset dK_{F,(p,g,t)}^1(T_{(p,g,t)}(V \times [0, 1]))$. We define

$$W_{(p,g)} = \{(v, w) \in T_{(p,g)}(\Delta \times G^J(R^S)) | v \in \mathcal{L}(p, g)^\perp, d\mathcal{L}_{(p,g)}(v, w) = 0\}.$$

For any $v \in \mathcal{L}(p, g)^\perp \cap T_p\Delta$, the $w \in T_gG^J(R^S)$ satisfying $d\mathcal{L}_{(p,g)}(v, w) = 0$ is determined uniquely and $(v, w) \in T_{(p,g)}V$ is satisfied, as seen in Claim 1 (v). Therefore, $W_{(p,g)}$ is an $M - k - 1$ -dimensional linear space in $T_{(p,g)}V$.

Since $dK_{F,(p,g,t)}^1$ maps to p^\perp , all we have to show for its surjectivity is that $dK_{F,(p,g,t)}^1(W_{(p,g)} \times 0)$ is an $M - k - 1$ -dimensional space such that $dK_{F,(p,g,t)}^1(W_{(p,g)} \times 0) \cap \mathcal{L}(p, g) = 0$.

For any $(v, w) \in W_{(p,g)}$, $v^T z^u(p) = 0$ because $v \in \mathcal{L}(p, g)^\perp$ and $z^u(p) = z^1(\mathcal{L}(p, g)) \in \mathcal{L}(p, g)$, where the equality is derived from $\tilde{K}_F^1(p, g, t) = K_F^1(p, g, t) = 0$. In addition, for any $(v, w) \in W_{(p,g)}$, $dK_{F,(p,g,t)}^1(v, w, 0) = dz^u(v) + tdF(d\mathcal{L}_{(p,g)}(v, w)) = dz^u(v)$ because $d\mathcal{L}_{(p,g)}(v, w) = 0$. However, $dz^u(v) \notin \mathcal{L}(p, g)$ because $dz^u(v) \notin v^\perp$ as shown in Claim 2. Thus, $dK_{F,(p,g,t)}^1(W_{(p,g)} \times 0) \cap \mathcal{L}(p, g) = 0$. For any independent v' and v'' in $T_p\Delta$, $dz_p^u(v')$ and $dz_p^u(v'')$ are independent because of the revealed axiom of the first consumer's strictly convex preference. Thus, $dK_{F,(p,g,t)}^1(W_{(p,g)} \times 0)$ is $M - k - 1$ -dimensional. ■

A.4 Proof of Lemma 4

Let $\{B_i\}_i$ be an open cover of $G_{++}^k(R^M)$: $G_{++}^k(R^M) \subset \bigcup_i B_i$. We first show that all we have to prove is the existence of a function $F^i : B^i \rightarrow R^M$ defined on each open subset satisfying Walras' law $F^i(L) \in L$, (i') $\tilde{z}^1(L^\lambda(r)) + t^\lambda(r)F^i(L^\lambda(r)) = 0$, and (ii') $(d\tilde{z}^1 + t^\lambda(r)dF^i)_{L^\lambda(r)}$ maps $\alpha_l^\lambda(r)$ to $\beta_l^\lambda(r)$, $l = 2, \dots, k$.

Suppose that we have obtained such F^i 's. Each F^i can be extended to a smooth function $\bar{F}^i : G_{++}^k(R^M) \rightarrow R^M$ satisfying $\bar{F}^i|B^i = F^i$ and $\bar{F}^i(L) \in L$. Let $\{\xi_i\}$ be a partition of unity subordinate to $\{B_i\}_i$, that is, $\xi_i : G_{++}^k(R^M) \rightarrow R$ is a smooth function such that $\sum_i \xi_i(L) = 1$, $0 \leq \xi_i(L) \leq 1$ for any L , and $\xi_i(L) = 0$ when $L \notin B_i$. Note that

for any L , $\sum_{i:L \in B^i} \xi_i(L) = 1$, where $\sum_{i:L \in B^i}$ denotes the sum over i 's such that $L \in B^i$. We define F as $F(L) = \sum_i \xi_i(L) \bar{F}^i(L)$.

It is clear that this F satisfies Walras' law. In addition, $\tilde{z}^1(L^\lambda(r)) + t^\lambda(r)F(L^\lambda(r)) = \sum_{i:L^\lambda(r) \in B_i} \xi_i(L^\lambda(r))(\tilde{z}^1(L^\lambda(r)) + t^\lambda(r)\bar{F}^i(L^\lambda(r))) = 0$ because $\bar{F}^i(L^\lambda(r)) = F^i(L^\lambda(r))$ when $L^\lambda(r) \in B_i$ and each F^i satisfies (i'); hence, (i) is satisfied.

To verify (ii), we first show $\sum_i \bar{F}^i(L^\lambda(r))d\xi_{i,L^\lambda(r)} = 0$. Note that $d\xi_{i,L^\lambda(r)} = 0$ for i such that $L^\lambda(r) \notin B_i$ because for such i , $\xi_i(L) = 0$ for any L in the neighborhood of $L^\lambda(r)$. Further, note that if (i') is satisfied, then $F^i(L^\lambda(r)) = \frac{\tilde{z}^1(L^\lambda(r))}{t^\lambda(r)}$, which is independent of i . Then, $\sum_i \bar{F}^i(L^\lambda(r))d\xi_{i,L^\lambda(r)} = \sum_{i:L^\lambda(r) \in B_i} F^i(L^\lambda(r))d\xi_{i,L^\lambda(r)} = \frac{\tilde{z}^1(L^\lambda(r))}{t^\lambda(r)} \sum_{i:L^\lambda(r) \in B_i} d\xi_{i,L^\lambda(r)} = 0$ because $\sum_{i:L^\lambda(r) \in B_i} \xi_i(L) = 1$ for any L in the neighborhood of $L^\lambda(r)$.

Applying this result with the F defined above, we have $(d\tilde{z}^1 + t^\lambda(r)dF)_{L^\lambda(r)} = d\tilde{z}_{L^\lambda(r)}^1 + t^\lambda(r) \sum_i (\bar{F}^i(L^\lambda(r))d\xi_{i,L^\lambda(r)} + \xi_i(L^\lambda(r))d\bar{F}_{L^\lambda(r)}^i) = d\tilde{z}_{L^\lambda(r)}^1 + t^\lambda(r) \sum_i \xi_i(L^\lambda(r))d\bar{F}_{L^\lambda(r)}^i = \sum_{i:L^\lambda(r) \in B_i} \xi_i(L^\lambda(r))(d\tilde{z}_{L^\lambda(r)}^1 + t^\lambda(r)dF_{L^\lambda(r)}^i)$. Thus, (ii) is satisfied when each F^i satisfies (ii').

We let $\{B_i\}$ be an open cover of $G_{++}^k(R^M)$ where each B_i is sufficiently small. Let us pick a B_i and find $F^i : B^i \rightarrow R^M$ satisfying Walras' law, (i'), and (ii'). If B_i includes no budget set belonging to the paths, that is, if $L^\lambda(r) \notin B_i$ for any λ and r , then (i') and (ii') are insignificant with respect to F^i . It is easy to find F^i satisfying Walras' law.

Let $L^\lambda(r) \in B_i$ for some λ and r . Since B_i can be sufficiently small, without loss of generality, we assume that B_i includes no budget set belonging to other paths. We let $\phi : B_i \rightarrow R^{(M-k)k}$ be the coordinate function such that $\phi(L^\lambda(r)) = (r, 0, \dots, 0) = re_1$ for $L^\lambda(r) \in B_i$, where $e_1 \in R^{(M-k)k}$ is the vector whose first element is 1 and the other elements are 0's.

Note that (i') restricts the value of F^i at $L^\lambda(r) \in B^i$ and (ii') restricts the derivative dF^i at $L^\lambda(r) \in B^i$. However, if (i') is satisfied, then $d\tilde{z}^1(dL_r^\lambda) + t^\lambda(r)dF^i(dL_r^\lambda) = -F^i(L^\lambda(r))dt_r^\lambda$ holds; hence, the derivative dF^i should satisfy

$$(d\tilde{z}^1 + t^\lambda(r)dF^i)_{L^\lambda(r)}(dL_r^\lambda) = -F^i(L^\lambda(r))dt_r^\lambda \quad (11)$$

in addition to (ii'). Using the Jacobian with the coordinate system, the conditions (11) and (ii') on $dF_{L(r)}$ can be summarized as

$$\begin{aligned} & \left[\frac{\partial(\tilde{z}^1 \circ \phi^{-1})}{\partial y} + t^\lambda(r) \frac{\partial(F^i \circ \phi^{-1})}{\partial y} \right] [e_1, d\phi(\alpha_2^\lambda(r)), \dots, d\phi(\alpha_k^\lambda(r))] \\ & = \left[-F^i(L^\lambda(r)) \frac{\partial t^\lambda}{\partial r}(r), \beta_2^\lambda(r), \dots, \beta_k^\lambda(r) \right], \end{aligned} \quad (12)$$

where the Jacobian on the left-hand side is evaluated at $(r, 0, \dots, 0) = re_1 \in R^{(M-k)k}$, $y = (y_1, \dots, y_{(M-k)k})$ denotes the coordinate of $R^{(M-k)k}$, and $d\phi$ is the derivative of ϕ at $L^\lambda(r)$. Note that $d\phi(dL_r^\lambda) = e_1$ because $\phi(L^\lambda(r)) = (r, 0, \dots, 0)$.

Recall that $L \in G_{++}^k(R^M)$ is a k -dimensional linear space in R^M . Since B_i can be sufficiently small, without loss of generality, we can assume that any vector $x = (x_1, \dots, x_M) \in R^M$ satisfying $x \in L$ for $L \in B_i$ can be determined uniquely from its first k elements $x = (x_1, \dots, x_k) \in R^k$ by relabeling the coordinates of R^M if needed. For a vector $x \in R^M$, we let $\check{x} = (x_1, \dots, x_k)$ denote the first k elements and $\dot{x} = (x_{M-k+1}, \dots, x_M)$ denote the last $M-k$ elements. Using these symbols, it can be assumed that there exists a function $X : B^i \rightarrow R^{(M-k)k}$ such that $X(L)$ is an $(M-k) \times k$ matrix and $\dot{x} = X(L)\check{x}$ if and only if $x \in L$ for $L \in B_i$.

We construct F^i as follows. First, we let $\Psi : R \rightarrow R^k$ be a smooth function satisfying

$$\check{z}^1(L^\lambda(r)) + t^\lambda(r)\Psi(r) = 0 \text{ for } r \in [0, 1] \text{ such that } L^\lambda(r) \in B_i. \quad (13)$$

In fact, we define $\Psi(r) = -\frac{\check{z}^1(L^\lambda(r))}{t^\lambda(r)}$ for $t^\lambda(r) > 0$. Since $t^\lambda(r) = 0$ when $\lambda = 1$ and $r = 0$, we define $\Psi(0) = \lim_{r \rightarrow 0} -\frac{\check{z}^1(L^\lambda(r))}{t^\lambda(r)}$ if the B_i includes the terminal point of the first path: $L^1(0) \in B_i$.

Second, we let $\Phi : R \rightarrow R^{k \times (M-k)k}$ be a smooth function such that

$$\begin{aligned} & \left[\frac{\partial(\check{z}^1 \circ \phi^{-1})}{\partial y} + t^\lambda(r)\Phi(r) \right] [e_1, d\phi(\alpha_2^\lambda(r)), \dots, d\phi(\alpha_k^\lambda(r))] \\ & = \left[-\Psi(r)\frac{\partial t^\lambda}{\partial r}(r), \check{\beta}_2^\lambda(r), \dots, \check{\beta}_k^\lambda(r) \right]. \end{aligned} \quad (14)$$

The existence of such an $\Phi(r)$ follows from the fact that $\Phi(r)$ is a $k \times (M-k)k$ matrix while (14) consists of only $k \times k$ equations and that the $(M-k)k \times k$ matrix $[e_1, d\phi(\alpha_2^\lambda(r)), \dots, d\phi(\alpha_k^\lambda(r))]$ is of full column rank by our choice of $\alpha_i^\lambda(r)$.

Finally, we define $F^i = (\check{F}^i, \dot{F}^i)$ as

$$\check{F}^i(L) = \Psi(\phi_1(L)) + \Phi^2(\phi_1(L))\phi_2(L) + \dots + \Phi^{(M-k)k}(\phi_1(L))\phi_{(M-k)k}(L), \quad (15)$$

and

$$\dot{F}^i(L) = X(L)\check{F}^i(L)$$

where $\phi_j(L)$ denotes the j -th element of $\phi(L)$ and $\Phi^j(r)$ denotes the j -th column vector of $\Phi(r)$.

We should observe that this F^i is a desirable function. It is clear that F^i satisfies Walras' law.

Since $\phi(L^\lambda(r)) = (r, 0, \dots, 0)$, $\check{F}^i(L^\lambda(r)) = \Psi(r)$; hence, $\check{z}^1(L^\lambda(r)) + t^\lambda(r)\check{F}^i(L^\lambda(r)) = 0$ is satisfied because of (13). Since $\check{z}^1(L^\lambda(r)) \in L^\lambda(r)$ and $F^i(L^\lambda(r)) \in L^\lambda(r)$, $\check{z}^1(L^\lambda(r)) + t^\lambda(r)\check{F}^i(L^\lambda(r)) = X(L^\lambda(r))(\check{z}^1(L^\lambda(r)) + t^\lambda(r)\check{F}^i(L^\lambda(r))) = 0$. Thus (i') is satisfied.

To verify (ii), we observe a relation between $\Psi(r)$ and $\Phi(r)$ satisfying (13) and (14). From (13), $d\check{z}^1(dL_r^\lambda) + t^\lambda(r)\frac{\partial\Psi}{\partial r}(r) = -\Psi(r)dt_r^\lambda$, that is,

$$\left[\frac{\partial(\check{z}^1 \circ \phi^{-1})}{\partial y} \right] e_1 + t^\lambda(r)\frac{\partial\Psi}{\partial r}(r) = -\Psi(r)\frac{\partial t^\lambda}{\partial r}(r).$$

Comparing this with (14), we have $\frac{\partial\Psi}{\partial r}(r) = \Phi(r)e_1 = \Phi^1(r)$.

Because of $\phi(L^\lambda(r)) = (\phi_{i1}(L^\lambda(r)), \phi_{i2}(L^\lambda(r)), \dots, \phi_{(M-k)k}(L^\lambda(r))) = (r, 0, \dots, 0)$, the derivative of (15) at $\phi_i(L^\lambda(r))$ with respect to the coordinate system is calculated as $\left[\frac{\partial\check{F}^i \circ \phi^{-1}}{\partial y_1} \right] = \frac{\partial\Psi}{\partial r}(r) = \Phi^1(r)$ and $\left[\frac{\partial\check{F}^i \circ \phi^{-1}}{\partial y_j} \right] = \Phi^j(r)$, $j = 2, \dots, (M-k)k$, that is, $\left[\frac{\partial\check{F}^i \circ \phi^{-1}}{\partial y} \right] = \Phi(r)$. Therefore, $(d\check{z}^1 + t^\lambda(r)d\check{F}^i)_{L^\lambda(r)}(\alpha_i^\lambda(r)) = \check{\beta}_i^\lambda(r)$ is satisfied because of (14).

Since $\check{z}^1(L) + tF^i(L) \in L$ for any t and $L \in B_i$, $(d\check{z}^1 + tF^i)_L$ maps to L at (L, t) satisfying $\check{z}^1(L) + tF^i(L) = 0$. In particular, $(d\check{z}^1 + t^\lambda(r)dF^i)_{L^\lambda(r)}$ maps to $L^\lambda(r)$. Hence, $(d\check{z}^1 + t^\lambda(r)dF^i)_{L^\lambda(r)}(\alpha_i^\lambda(r)) \in L^\lambda(r)$. The first k elements of this vector are $(d\check{z}^1 + t^\lambda(r)d\check{F}^i)_{L^\lambda(r)}(\alpha_i^\lambda(r))$ and the last $M-k$ elements of the vector are $(d\check{z}^1 + t^\lambda(r)d\check{F}^i)_{L^\lambda(r)}(\alpha_i^\lambda(r))$. we have shown $(d\check{z}^1 + t^\lambda(r)d\check{F}^i)_{L^\lambda(r)}(\alpha_i^\lambda(r)) = \check{\beta}_i^\lambda(r)$. Then, $(d\check{z}^1 + t^\lambda(r)d\check{F}^i)_{L^\lambda(r)}(\alpha_i^\lambda(r)) = X(L^\lambda(r))(d\check{z}^1 + t^\lambda(r)d\check{F}^i)_{L^\lambda(r)}(\alpha_i^\lambda(r)) = X(L^\lambda(r))\check{\beta}_i^\lambda(r) = \dot{\beta}_i^\lambda(r)$ because $\beta_i^\lambda(r) \in L^\lambda(r)$ from our choice. Thus, (ii') is satisfied.

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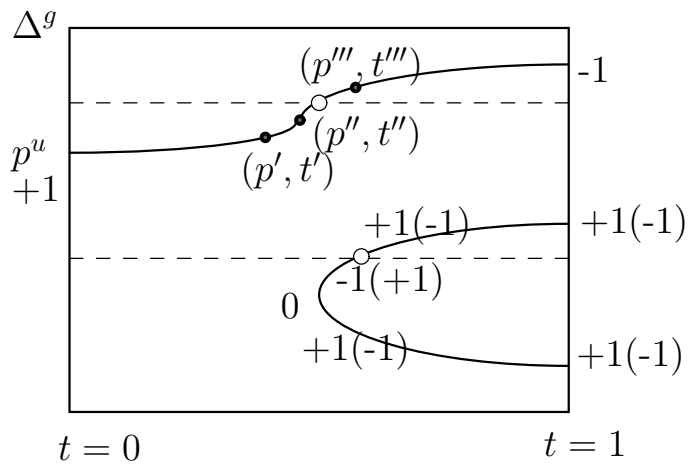


Figure 1: Example of homotopy paths