# The index of a GEI economy when the degree of incompleteness is odd

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### Abstract

Momi (2003b) proved that the index of an economy with incomplete real asset markets is typically +1 when the degree of incompleteness, defined as the difference between the number of states and the number of securities, is even. This paper considers the case where the degree of incompleteness is odd and proves that any odd number can be the index of an economy of this type.

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## 1 Introduction

The index theorem was first introduced to economics by Dierker (1972) with application to the Arrow–Debreu exchange economy. Though the index theorem is a mathematical concept, it has important economic implications including, for example, the existence of equilibrium, the number of equilibria, and the local uniqueness of equilibrium (Mas-Colell et al. 1995). However, a particular feature of the index theorem is its power to detect multiple equilibria. The index theorem maintains that the sum of indices at equilibrium over all equilibria in an economy, that is, the index of the economy, equals +1. Therefore, the existence of equilibrium with an index value of -1 implies that the economy has at least two other equilibria with an index value of +1.

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The general equilibrium model with incomplete markets (GEI model) is an extension of the Arrow–Debreu model, in that it more precisely describes the trading mechanism for uncertainty. Therefore, it is a natural question to ask whether the index theorem holds in the GEI model. Momi (2003b) provides a partial response: the index theorem holds when the degree of incompleteness, defined as the difference between the number of states and the number of securities, is even. See Predtetchinski (2003) for another approach to this result, where it is transparently shown why enen degree of the incompleteness should be required for the index theorem. The main purpose of this paper is to study the alternative where the degree of incompleteness is odd. Combined together, this paper and Momi (2003b) then complete the index theorem for the GEI model. The overall result of this paper is striking. We show that any odd number can be the index of a GEI economy with an odd degree of incompleteness. That is, the index theorem does not hold for the GEI economy of this type. This lies in sharp contrast to Dierker's index theorem for the Arrow–Debreu economy.

It is well known that the index is homotopy invariant. In actuality, a simple proof of the index theorem for the Arrow–Debreu economy is a byproduct of the homotopy path following method for computation of equilibrium, given in Garcia and Zangwill (1981), for example. Though Brown et al. (1996) and DeMarzo and Eaves (1996) have presented homotopy methods to compute an equilibrium in the GEI economy, they have not presented the index result directly as in the Arrow–Debreu economy. The problem is discontinuity of demand functions at "bad prices" where the budget set drops rank. In the former, the homotopy paths thus have "bad points" where the paths are not continuous. In the latter, while discontinuity of the homotopy paths disappears through introduction of the Grassmanian manifold, the index is not well defined because the Grassmanian manifold is not generally orientable.

Though the homotopy paths in Brown et al. (1996) are not continuous at the bad points, Momi (2003b) proved the rule governing the index change at the discontinuous bad points. According to the index change rule, no index change occurs when the degree of incompleteness is even, thus the index invariant property is sustained regardless of the discontinuity of the homotopy paths. The index theorem holds. On the other hand, if the degree of incompleteness is odd, the index change at a bad point can occur. Whether it occurs then depends on how the homotopy path intersects the bad point. More precisely speaking, it depends on the direction the path passes through the bad point.

When an economy is given, the homotopy paths of Brown et al. (1996) are determined. Then, by applying the index change rule of Momi (2003b), we can compute the index of the economy. In this paper, we proceed by the converse. Roughly speaking, when an odd number is given, we first draw a picture of the paths so that it induces the given odd number as the index when the index change rule is applied. We then construct an economy so that the economy actually produces the paths as its homotopy paths. The economy will then have the given odd number as its index. Let us consider the problem in more detail.

Our first task is to draw a picture of the paths so that the paths induce the given odd number as the index when the index change rule is applied. As discussed, whether the index change at a bad point on a path occurs depends only on how the path intersects the discontinuous point. The total space in which the path is drawn is the product of the price set and the homotopy parameter set. The set of bad prices is the null set in the price set. Therefore, when a target bad point is fixed, we can draw a path in almost any direction passing through the bad point. Therefore, it is not difficult to draw a path that induces the index change.

Our second task is to construct an economy that induces the given paths as its homotopy paths. For arbitrarily drawn paths, is there an economy that realizes the paths as its homotopy paths? This is the characterization problem of the homotopy paths of an economy. Roughly speaking, we show that any reasonable paths can be realized by an economy. Note that the characterization problem of homotopy paths is closely related to the characterization problem of the equilibrium price set. In fact, they coincide at the terminal homotopy parameter value where the homotopy function is the aggregate excess demand function. In the case of the Arrow–Debreu economy, the characterization problem of equilibrium price set is addressed by Mas-Colell (1977), as an almost straight application of the Sonnenchein–Mantel–Debreu Theorem. In the case of the incomplete market economy, a result similar to Sonnenchein–Mantel–Debreu is found in Momi (2003a). However, it does not deal with the characterization of the excess demand function around bad prices. On the other hand, to construct homotopy paths that induce an index other than +1, we need at least one path to pass through a bad point. Therefore, we must control exactly the behavior of the excess demand around bad prices. One technical difficulty faced in this paper is how to fill this gap.

In Section 2, we explain the homotopy path characterization problem in the Arrow–Debreu economy and show that Mas-Colell's theorem on the characterization of equilibrium set plays a crucial role in its solution. This section provides a guide to understanding the paper's fundamental logic. Section 3 describes the setup of the GEI economy. Section 4 provides an answer to the equilibrium set characterization problem in the GEI model, that is, we prove the GEI model version of Mas-Colell's theorem. Section 5 reviews Momi's (2003b) index change rule. The main result of the paper, that is, the existence of an economy with an arbitrary odd index number, is proved in the final section.

## 2 Sketch for the Arrow–Debreu economy

In this section, we explain our methodology, taking the Arrow–Debreu economy as an example. This section is quite useful for understanding the paper's logic.

### 2.1 Homotopy, index, and index rule

We first review the homotopy path and Dierker's index theorem in the Arrow–Debreu economy. We consider the Arrow–Debreu exchange economy with $M$ commodities and $I$ consumers, where each consumer $i$ is represented by a pair $(\precsim^i, \omega^i)$ of a strictly convex, monotone, continuous, complete preference ordering $\precsim^i$ on the consumption set $R^M_+$ and an initial endowment $\omega^i \in R^M_{++}$. Let $\Delta = \{p \in R^M_{++} | \parallel p \parallel = 1\}$[1] be the price set and $z^i : \Delta \to R^M$ be the excess demand function of consumer $i$: $z^i(p) = \{x - \omega^i \in R^M | p(x - \omega^i) = 0$ and $x' \precsim^i x$ for any $x' \in R^M_+$ satisfying $p(x' - \omega^i) = 0\}$.

We let $Z = \sum_i z^i$ denote the aggregate excess demand function of all consumers and $Z^c = \sum_{i \geq 2} z^i$ denote the aggregate excess demand function of all consumers, except the first consumer. Because of Walras' law: $pZ(p) = 0$, only $M - 1$ elements of $Z = (Z_1, \dots, Z_M)$ are independent and the equilibrium price set is defined by $\hat{Z}^{-1}(0)$ where " ˆ " denotes to drop the last element, that is, $\hat{Z} = (Z_1, \dots, Z_{M-1})$. The homotopy $H : \Delta \times [0, 1] \to R^M$ we deal with is $H(p, t) = z^1(p) + tZ^c(p)$ where $t \in [0, 1]$ is the homotopy parameter. Note that $H(p, 1) = Z(p)$, $H(p, 0) = z^1(p)$ and the supporting price $p^u$ of the first consumer's indifference surface at $\omega^1$ is the solution of $H(\cdot, 0) = 0$.

Let us list the properties of $H^{-1}(0)$ satisfied for almost all $\omega = (\omega^1, \dots, \omega^I) \in R^{MI}_{++}$ when each $\precsim^i$ is represented by a smooth utility function $u^i : R^M_+ \to R$. The following, (C1)–(C4), are well known. They have been demonstrated in many works, for example, in Mas-Colell et al. (1995).

- (C1) $H^{-1}(0)$ is a one-dimensional smooth compact submanifold of $\Delta \times [0, 1]$;
- (C2) $H^{-1}(0)$ is bounded away from the boundary $\partial\Delta \times [0, 1]$;[2]
- (C3) $H^{-1}(0)$ is not tangential to the boundary $\Delta \times \{0, 1\}$;
- (C4) $p^u$ is the unique solution of $H(\cdot, 0) = 0$.

We denote the one-dimensional smooth connected submanifold as path and path in

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[1] $\| \cdot \|$ denotes the Euclid norm.

[2] $\partial X$ denotes the boundary of $X$.

$H^{-1}(0)$ as homotopy path. The properties (C1)–(C4) imply that the homotopy paths are typically drawn as in Figure 1: one path, whose one end point is $(p^u, 0)$ has another end point $(p^*, 1)$ on $\Delta \times \{1\}$; each of other paths has both its end points on $\Delta \times \{1\}$.

Suppose $(p, t)$ and $(p, t')$ are in $H^{-1}(0)$. This implies $z^1(p) + tZ^c(p) = 0$ and $z^1(p) + t'Z^c(p) = 0$. Then $t - t' = 0$ holds or $p = p^u$ and $Z^c(p^u) = 0$ holds. The latter case is, however, not generically satisfied because $Z^c(p^u) = 0$ is not robust to perturbations of $\{\omega^i\}_{i \neq 1}$. Thus, we have

(C5) if $(p, t)$ and $(p, t')$ are in $H^{-1}(0)$, then $t = t'$.

This implies that the projection of $H^{-1}(0)$ onto $\Delta$ is also a one-dimensional manifold and it does not have a cross.

When (C1) is satisfied, we can parameterize each path in $H^{-1}(0)$ as $(p(s), t(s)) \in \Delta \times [0, 1]$ by a parameter $s$ and smooth functions $s \to p(s)$ and $s \to t(s)$. Without loss of generality, we assume that $(\frac{\partial p}{\partial s}(s), \frac{\partial t}{\partial s}(s)) \neq (0, 0)$ at any $s$ for the parameterization. Since $z^1(p(s)) + t(s)Z^c(p(s)) = 0$ holds for any $s$, by taking the derivative, we obtain

$$\left[\frac{\partial z^1}{\partial p} + t\frac{\partial Z^c}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] + Z^c\frac{\partial t}{\partial s} = 0.^3$$

If $\left[\frac{\partial p}{\partial s}(s)\right] = 0$ at some $s$, then $Z^c(p(s)) = 0$ because $\frac{\partial t}{\partial s}(s) \neq 0$ from the assumption of our parameterization. Then $z^1(p(s)) = 0$, hence $p(s) = p^u$. However $Z^c(p^u) = 0$ is not robust to perturbations of $\{\omega^i\}_{i \neq 1}$. Therefore each path satisfies $\left[\frac{\partial p}{\partial s}(s)\right] \neq 0$, that is,

(C6) the price $p(s)$ changes according to infinitesimal movements along the path.

The index at an equilibrium $\bar{p} \in \hat{Z}^{-1}(0)$ is, for example, defined by $index\hat{Z}(\bar{p}) = (-1)^{M-1} sign\left(det\left[\frac{\partial \hat{Z}}{\partial \hat{p}}(\bar{p})\right]\right)$.[4] The index of an economy is defined by the sum of these indices at equilibrium over all equilibria: $\sum_{\bar{p} \in \hat{Z}^{-1}(0)} index\hat{Z}(\bar{p})$. Therefore, the index is well defined when the equilibrium set is finite and the aggregate excess demand function is differentiable at each equilibrium. Dierker's index theorem argues that the index of an economy is +1 for almost all $\omega$.

A simple proof of Dierker's index theorem is as follows. (See Garcia and Zangwill (1981), Mas-Colell (1985), or Mas-Colell et al. (1995) for details.) In the same way as we defined index at equilibrium $p \in \hat{Z}^{-1}(0)$, we define the index of the homotopy $H$ at $(\bar{p}, \bar{t}) \in H^{-1}(0)$ as $index\hat{H}(\bar{p}, \bar{t}) = (-1)^{M-1} sign\left(det\left[\frac{\partial \hat{H}}{\partial \hat{p}}(\bar{p}, \bar{t})\right]\right)$.

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[3] For a function $f : R^m \to R^n$, $f(x) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$, we let $\frac{\partial f}{\partial x}(\bar{x})$ denote the derivative of $f$ at $\bar{x}$. When the point $\bar{x}$ is clear, we simply write $\frac{\partial f}{\partial x}$. We write $\left[\frac{\partial f}{\partial x}\right]$ to highlight that it is the $n \times m$ Jacobian matrix. We use the symbol 0 to denote any dimensional zero matrices.

[4] $\frac{\partial \hat{Z}}{\partial \hat{p}}$ denotes the derivative of $\hat{Z} = (Z_1, \ldots, Z_{M-1})$ with respect to $\hat{p} = (p_1, \ldots, p_{M-1})$.

It is well known that when each $\precsim^i$ is represented by a smooth utility function, the index of the homotopy $H$ at each $(\bar{p}, \bar{t}) \in H^{-1}(0)$ is well defined for almost all $\omega$, and the following, (R1) and (R2), hold.

(R1) The index of $H$ at $(p^u, 0) \in H^{-1}(0)$ is +1.

(R2) Regularity of $H$: the $(M-1) \times M$ matrix $[\frac{\partial \hat{H}}{\partial \hat{p}}, \frac{\partial \hat{H}}{\partial t}]$ has rank $M-1$ at all $(p, t) \in H^{-1}(0)$.

It is also well known that (R2) implies the rule that determines the relation between indices at two different points on a homotopy path.

(R2') When we follow a homotopy path in $H^{-1}(0)$, index of $H$ is unchanged as long as we move in the same direction with respect to $t$ and is changed when the direction is changed.

We call (R1) and (R2') as the index rule for the Arrow–Debreu economy. According to this rule, index of $H$ at $(p^*, 1)$ is +1. Other paths have both end points on $t = 1$, and have indices +1 and -1 at their end points. Thus these indices sum up to +1 on $\Delta \times \{1\}$. Thus, Dierker's index theorem is proved.

## 2.2 Homotopy path characterization problem

In the previous section, we reviewed the properties of the homotopy paths and the index rule when an economy $\{\precsim^i, \omega^i\}_{i=1}^I$ is given. In this section, we ask the converse. When candidate paths in the space $\Delta \times [0, 1]$ satisfying (C1)–(C6) and candidate index number consistent with the index rule are given, is there an economy $\{\precsim^i, \omega^i\}_i$ of a finite number of consumers that realizes the paths and the index as its homotopy paths and its index? The answer to this question follows.

Since the property (C4) depends on the characteristics of the first consumer, we assume that paths in $\Delta \times [0, 1]$ satisfying (C1)–(C6) are given after the first consumer is fixed.

**Theorem 1.** We let $M \geq 3$.[5] Suppose a (first) consumer $(\precsim^1, \omega^1)$, who has a smooth excess demand function $z^1$, is arbitrarily given and suppose paths in $\Delta \times [0, 1]$ satisfying (C1)–(C6) are arbitrarily given. Further, suppose index numbers (+1 or -1) consistent with the index rule (R1)(R2') are arbitrarily given at the end points of the paths. There

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[5] When $M = 2$, we need an additional assumption: the index values +1 and -1 should appear in turn on the one-dimensional price set $\Delta$.

then exists an economy $\{\precsim^i, \omega^i\}_i$, whose homotopy paths and the indices at the end points of each homotopy paths coincide with the given paths and the given index numbers.

We do not give a proof of this theorem, because we later prove a more general result in the set up of the GEI model. Here we provide a brief sketch of the proof, which should be a good guide for the main result. In the sketch, it should be highlighted that Mas-Colell's theorem on the characterization of the equilibrium price set (see below) plays a key role.

Let $\Delta_\epsilon = \{p \in \Delta | p_l \geq \epsilon \text{ for all } l = 1, \ldots, M\}$ for $\epsilon > 0$. For any function $z : \Delta \to R^M$, let $E_z = \{p \in \Delta | z(p) = 0\}$. That is, if $z$ is the aggregate excess demand function of an economy, then $E_z$ denotes the equilibrium price set of the economy.

**Theorem 2.** [Mas-Colell (1977)] Let $z : \Delta \to R^M$ be a continuous function such that (W) $z$ satisfies Walras' law: $pz(p) = 0$, (BB) $z$ is bounded from below: there is $\alpha \in R$ such that $z(p) > \alpha \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)$, and (BC) $z$ satisfies the boundary condition: $\| z(p) \| \to \infty$ as $p \to \bar{p} \in \partial\Delta$.

Then, for any sufficiently small $\epsilon > 0$, there exists an economy $\{\precsim^i, \omega^i\}_{i=1}^M$ whose aggregate excess demand function $z^*$ satisfies (i) $z^*|\Delta_\epsilon = z|\Delta_\epsilon$, (ii) $E_{z^*} = E_z \subset \Delta_\epsilon$, and (iii) $\sum_{l=1}^M z_l^*(p) > 0$ for $p \in \Delta \backslash \Delta_\epsilon$.

**Sketch of the proof of Theorem 1.** Without loss of generality, we assume that $r$ paths are given; each path is parameterized as $(p^j(s), t^j(s))$ by $s \in [0, 1]$, $j = 1, \ldots, r$; the first path $(p^1(s), t^1(s))$ has its end points in both of $\Delta \times \{0\}$ and $\Delta \times \{1\}$: $(p^1(0), t^1(0)) = (p^u, 0)$ and $t^1(1) = 1$; each of the other paths, $j \geq 2$, has both its end points on $\Delta \times \{1\}$: $t^j(0) = t^j(1) = 1$. We also assume that candidate index numbers at one end point $(p^j(0), t^j(0))$ of paths $j \geq 2$ are given. The index numbers at the other end points must be determined by the index rule.

STEP 1: We construct a candidate aggregate excess demand of consumers $i \geq 2$. For any function $F : \Delta \to R^M$, we define $H_F : \Delta \times [0, 1] \to R^M$ as $H_F(p, t) = z^1(p) + tF(p)$ and define the index of $H_F$ analogously to that of $H$. We construct a continuous function $F : \Delta \to R^M$ such that (1-1) $F$ satisfies Walras' law: $pF(p) = 0$; (1-2) each $(p^j(s), t^j(s))$ is a regular homotopy path of $H_F(p, t) = 0$; and (1-3) index of $H_F$ at each of $(p^j(0)t^j(0))$, $j \geq 2$, is the given index number.

STEP 2: We modify $F$ to $Z$ so that (2-1) $Z$ equals to $F$ in a neighborhood of $\bigcup_{j,s} p^j(s)$; (2-2) $H_Z(p, t) = 0$ does not have solutions other than $(p^j(s), t^j(s))$ and (2-3) $Z$ is a continuous function satisfying (W)(BB)(BC) in Mas-Colell's theorem.

STEP 3: By applying Mas-Colell's theorem to the function $Z$, we construct an economy $\{\precsim^i, \omega^i\}_{i=2}^{M+1}$ of $M$ consumers, whose aggregate excess demand function $Z^*$ then satisfies the statements (i)–(iii) in the theorem.

When $\epsilon$ is sufficiently small, $\sum_{l=1}^{M} z_l^1(p) > 0$ for $p \in \Delta \backslash \Delta_\epsilon$. This and (iii) implies that $H_{Z^*}(p,t) = z^1(p) + tZ^*(p) = 0$ has no solution in $(\Delta \backslash \Delta_\epsilon) \times [0,1]$. Because of (i) and (2-2), $(p^j(s), t^j(s))$ are all the solutions of $H_{Z^*}(p,t) = 0$ and because of (2-1), $index \hat{H}_{Z^*} = index \hat{H}_Z = index \hat{H}_F$ at each end point of the paths. Thus $\{(\precsim^i, \omega^i)\}_{i=1}^{M+1}$ is the economy that realizes the given paths and the given index numbers at end points of the paths. ■

# 3 GEI model

The model we deal with is a standard two-period economy with incomplete real asset markets ($S$ possible states in period 1; $N$ goods in each state, so that $R^M$, $M = (S+1)N$, with period 0 as state 0, is the total commodity space; $J (< S)$ real assets $A^j$, $j = 1, \ldots, J$, each of which promises the delivery of a bundle of commodities $A_s^j = (A_{s1}^j, \ldots, A_{sN}^j)$ if state $s \in \{1, \ldots, S\}$ occurs in period 1). We represent the asset structure by $A = \{A_{sn}^j\}$. Throughout the paper, we assume that redundant assets are excluded. As in Section 2, each consumer indexed by $i$ is defined by $(\precsim^i, \omega^i)$ where $\precsim^i$ is his/her strictly convex, monotone, continuous, complete preference ordering on the consumption set $R_+^M$ and $\omega^i \in R_{++}^M$ is his/her initial endowment. We let $p$, $x^i$, $\omega^i$, and $z^i = x^i - \omega^i$, respectively, denote price, consumption, endowment, and excess demand of consumer $i$, where $p = (p_0 \ldots, p_S)$, $p_s = (p_{s1}, \ldots, p_{sN})$ and so on. We often relabel $p$ as $p = (p_1, \ldots, p_l, \ldots, p_M)$ where $p_1 = p_{01}, p_2 = p_{02}, \ldots, p_M = p_{SN}$ and relabel $x^i$, $\omega^i$ and $z^i$ in the similar way.

The budget set, on which the excess demand vector $z = (z_0, \ldots, z_S) \in R^M$ lies, is

$$L(p) = \{z \in R^M | pz = 0, p_{\mathbf{1}} \Box z_{\mathbf{1}} \in \langle A(p) \rangle\},$$

where $p_{\mathbf{1}} = (p_1, \ldots, p_S)$, $z_{\mathbf{1}} = (z_1, \ldots, z_S)$, $p_{\mathbf{1}} \Box z_{\mathbf{1}} = (p_1 z_1, \ldots, p_S z_S)^T$, $A(p)$ is the $S \times J$ payoff matrix with $p_s A_s^j$ as the $(s,j)$-element, $\langle A(p) \rangle$ is the subspace of $R^S$ spanned by the column vectors of $A(p)$, and $p = (p_0, \ldots, p_S) \in \Delta = \{p \in R_{++}^M | \parallel p \parallel = 1\}$ is the present value price system.[6] See Duffie and Shafer (1985) for justification of this definition of the budget set using the present value price system. The excess demand function of consumer $i$ is, then, defined by $z^i(p) = \{x - \omega^i \in R^M | (x - \omega^i) \in L(p)$ and $x' \precsim^i x$ for any $x' \in R_+^M$ satisfying $(x' - \omega^i) \in L(p)\}$.

A difficulty of the GEI model is that the excess demand function $z^i$ is not continuous at a price where the budget set $L(p)$ drops dimension, that is, where the payoff matrix $A(p)$ drops its rank. We call such a price "bad". We define $\Delta^b \equiv \{p \in \Delta | rank A(p) < J\}$

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[6] $x^T$ denotes the transpose of a vector or a matrix $x$.

as the set of bad prices and $\Delta^g \equiv \{p \in \Delta | rank A(p) = J\} = \Delta \backslash \Delta^b$ as the set of "good" prices. Note that, for $p \in \Delta^g$, the budget set $L(p)$ is a $k \equiv M - (S - J) - 1$-dimensional subspace of $R^M$, that is, $L(p)$ is an element of the Grassmanian $G^k(R^M)$. Since $L(p)$ satisfies the no-arbitrage condition, $L(p)$, $p \in \Delta^g$, is an element of $G^k_{++}(R^n) \equiv \{L \in G^k(R^n) | L \bigcap R_+ = \{0\}\}$.

# 4 Equilibrium characterization in GEI model

In Section 2, we found that Mas-Colell's theorem plays a crucial role when constructing a desirable economy in the Arrow–Debreu model. In this section, we prove the GEI model version of this theorem.

## 4.1 Properties of aggregate excess demand function

We first list the properties of the aggregate excess demand function $p \mapsto z(p)$ when all consumers maximize their utility subject to their net trade lying on $L(p)$, that is, $z$ is the sum of $z^i$ defined in Section 3. The following, (W),(H), and (BB), are clear from the definition.

(W) Walras' law: $z(p) \in L(p)$,

(H) Homogeneity: $z(p) = z(p')$ if $L(p) = L(p')$,

(BB) Bounded below: there is $\alpha \in R$ such that $z(p) > \alpha \mathbf{1}$
where $\mathbf{1} = (1, \dots, 1) \in R^M$.[7]

As for the continuity, it is well known that $z$ is continuous on $\Delta^g$ and not continuous on $\Delta^b$. Suppose two sequences of good prices converge to bad prices respectively, and the corresponding budget sets converge to the same $k$-dimensional subspace. It is then obvious that the corresponding excess demands converge to the same value. Therefore,

(C) Continuity: $z$ is continuous on $\Delta^g$, and if $p \to \bar{p} \in \Delta^b$, $p' \to \bar{p}' \in \Delta^b$ and
$lim_{p \to \bar{p}} L(p) = lim_{p' \to \bar{p}'} L(p')$, then $lim_{p \to \bar{p}} z(p) = lim_{p' \to \bar{p}'} z(p)$.

In this statement, for example, $p \to \bar{p} \in \Delta^b$ denotes that there is a sequence $\{p^n\}$ of good prices $p^n \in \Delta^g$ and it converges to a bad price $\bar{p} \in \Delta^b$ as $n \to \infty$. Note that, for $\bar{p} \in \Delta^b$, $lim_{p \to \bar{p}} L(p)$ is an element of $G^k_{++}(R^M)$ while $L(\bar{p})$ is not.

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[7] Note that $z^i(p) = \bar{z}^i(\pi^i(p))$ where $z^i$ is the excess demand of consumer $i$ in the GEI model, $\bar{z}^i$ is his/her excess demand in the Arrow–Debreu model and $\pi^i(p) \in \Delta \bigcap L(p)^\perp$ is the supporting price of his/her indifference surface at $z^i(p) + \omega^i$. Thus $z^i$ inherits the bounded below property, which is well known for the excess demand in the Arrow–Debreu model.

In the Arrow–Debreu economy, the boundary condition is written as $\| z(p) \| \to +\infty$ when $p \to \bar{p} \in \partial\Delta$. This is because the budget set $L(p) \in G_{++}^{M-1}(R^M)$ of the Arrow–Debreu model converges to $\bar{L} \in \partial G_{++}^{M-1}(R^M)$ if and only if $p$ converges to $\bar{p} \in \partial\Delta$. In the GEI model, the former is a necessary but not a sufficient condition for the latter. Therefore, the boundary condition is written as

(BC) Boundary condition: if ($p \to \bar{p} \in \partial\Delta$ and) $L(p) \to \bar{L} \in \partial G_{++}^{k}(R^M)$,
then $\| z(p) \| \to +\infty$.[8]

We define $G_{++} = \overline{L(\Delta^g)} \bigcap G_{++}^{k}(R^M)$.[9] It is convenient to rephrase the above properties of the aggregate excess demand function $z$ on $\Delta^g$ with properties of a function $\tilde{z}$ defined on $G_{++}$. It is not too difficult to show that the existence of a function $z : \Delta^g \to R^M$ satisfying (W)(H)(BB)(C) is equivalent to the existence of a continuous function $\tilde{z} : G_{++} \to R^M$ such that

(W') $\tilde{z}(L) \in L$,

(BB') there is $\alpha \in R$ such that $\tilde{z}(L) > \alpha\mathbf{1}$,

(BC') if $L \to \bar{L} \in \partial G_{++}^{k}(R^M)$, then $\| \tilde{z}(L) \| \to +\infty$.

Clearly, the continuity of $\tilde{z}$ implies the continuity (C) of $z$ and that $\tilde{z}$ is a function implies the homogeneity (H) of $z$. It is clear that $\tilde{z}$ should be interpreted as the aggregate excess demand function when consumers maximize their utilities subject to their net trades lying on $L \in G_{++}$. Note that we require the continuity of $\bar{z}$ on $\overline{L(\Delta^g)}$ for the second property of (C) of $z$. The continuity of $\bar{z}$ just on $L(\Delta^g)$ is insufficient.

## 4.2 Decomposition of aggregate excess demand function

Mas-Colell's theorem in the Arrow–Debreu economy represents an almost direct application of the Sonnenchein–Mantel–Debreu Theorem. In the case of the GEI economy, we do not have an exact counterpart to the S–M–D Theorem. Momi (2003a) showed that any function $p \to z(p)$ satisfying (W)(H)(C) can be decomposed into the sum of individual excess demand functions of rational agents on any compact good price set $P \subset \Delta^g$.

**Theorem 3.** [Momi (2003a)] Let $P \subset \Delta^g$ be a compact good price set. For any function $p \mapsto z(p)$ satisfying (W)(H)(C) there exists an economy $\{\precsim^i, \omega^i\}_i$ of a finite number of consumers whose aggregate excess demand is $z(p)$ for $p \in P$.

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[8] Remember $z^i(p) = \bar{z}^i(\pi^i(p))$ in the previous footnote. It is not difficult to observe $\pi^i(p) \to \bar{\pi} \in \partial\Delta$ when $L(p) \to \bar{L} \in \partial G_{++}^{k}(R^M)$.

[9] For a function $f : X \to Y$ and $W \subset X$, $f(W) = \{f(x) | x \in W\} \subset Y$. $\overline{X}$ denotes the closure of $X$.

Note that although this theorem provides the characterization of the aggregate excess demand function on any compact good price set, it does not deal with a characterization near bad prices. There is clearly a serious gap between this theorem, and the equilibrium price set characterization of an economy, because the economy might have equilibrium near a bad price. For our purpose, we need a characterization of the excess demand on any compact price set, as the S–M–D Theorem assures for the Arrow–Debreu economy. Though this is a difficult open problem, we can evade this difficulty by a special type of the GEI model, where the drop of the budget set is one-dimensional.

**Definition.** If the dimension of $L(p)$ is $k$ or $k-1$ for any price $p \in \Delta$, that is, if the rank of $A(p)$ is $J$ or $J-1$ for any price $p \in \Delta$, we say that the asset structure satisfies rank condition.

Hereafter, we consider the GEI model where the rank condition is satisfied. For example, consider an asset structure consisting of two assets ($J = 2$) with positive payoffs ($A^j_{sn} > 0$). It is clear that the dimension of $L(p)$ is, then, $k$ or $k-1$ for any $p$. When the rank condition is satisfied, we can prove the next proposition, or its equivalent corollary.

**Proposition 1.** Let the dimension of $L(p)$ be $k$ or $k-1$ for any $p \in \Delta$. Let $P \subset \Delta$ be a compact price set. For any function $p \rightarrow z(p)$ satisfying (W)(H)(C), there exists an economy $\{\precsim^i, \omega^i\}_i$ of a finite number of consumers whose aggregate excess demand is $z(p)$ for $p \in P \bigcap \Delta^g$.

**Corollary.** Let the dimension of $L(p)$ be $k$ or $k-1$ for any $p \in \Delta$. Let $G \subset G_{++} \equiv \overline{L(\Delta^g)} \bigcap G^k_{++}(R^M)$ be a compact set. For any continuous function $L \rightarrow z(L)$ satisfying (W'), there exists an economy $\{\precsim^i, \omega^i\}_i$ of a finite number of consumers whose aggregate excess demand is $z(L)$ for $L \in G$ when consumers maximize their utilities subject to their net trades lying on $L$.

In the rest of this section, we provide a proof of this proposition. Since the proof relies heavily on Momi (2003a), we first provide a sketch of Momi's (2003a) proof and show why the characterization of the aggregate excess demand near bad prices cannot be directly obtained.

In the projection-based approach of Momi (2003a), the decomposition is made using individual excess demands in the form of $L \rightarrow \beta(L) proj_L(\mathcal{A})$ where $\beta$ is a positive-valued, continuous function, $\mathcal{A}$ is a set defined by $\{x \in R^M | U(x) \geq \bar{U}\}$ where $\bar{U} > 0$ and $U$ is a Cobb–Douglas function $U(x) = (x_1)^{\alpha_1} \cdots (x_M)^{\alpha_M}$ with positive parameters such that

$\sum \alpha_l = 1$,[10] and $proj_L(\mathcal{A})$ denotes the point on $L$ closest to $\mathcal{A}$. We let $\Omega$ denote the set of $\mathcal{A}$'s with various $\bar{U}$ and $U$. For $L \in G^k_{++}(R^M)$ and $\mathcal{A} \in \Omega$, we let $a(L, \mathcal{A})$ denote the point in $\mathcal{A}$ closest to $L$ and let $b(L, \mathcal{A})$ denotes the point in $L$ closest to $\mathcal{A}$: $b(L, \mathcal{A}) \equiv proj_L(\mathcal{A})$.

The key for the proof of Theorem 3 is the following lemma.

**Lemma 1.** [Momi (2003a)] Let $\beta : G^k_{++}(R^M) \to R_{++}$ be a positive-valued continuous function and $\mathcal{A} \in \Omega$. Suppose they satisfy

$(*)$ if $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$, then $\beta(L) = \beta(L')$.

Then, there exists a consumer $(\precsim, \omega)$ whose excess demand function is $\beta(L) proj_L(\mathcal{A})$ when he/she maximizes his/her utility subject to his/her net trade lying in $L \in G^k_{++}(R^M)$.

It is clear that the condition $(*)$ is a necessary condition for a consumer's excess demand function to be $\beta(L) proj_L(\mathcal{A})$. If the condition $(*)$ is not satisfied, then the strict concavity of the consumer's preference ordering is violated.

However, as long as we focus on budget sets in $L(\Delta^g)$, the condition $(*)$ is not restrictive. Bottazzi and Hens (1996) proved that for any $L$ and $L'$ in $L(\Delta^g)$, and any $\mathcal{A} \in \Omega$, if $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$, then $L = L'$. Therefore, as long as we consider the budget sets of the GEI model at good prices, we do not have to consider the condition $(*)$. Thus Momi (2003a) could arbitrarily decompose the candidate function $L \to z(L)$ to functions in the form of $\beta(L) proj_L(\mathcal{A})$ and construct each consumer realizing each $\beta(L) proj_L(\mathcal{A})$ as his/her excess demand.

This is, however, not the case for $\overline{L(\Delta^g)}$. It is possible that two sequences of good prices converge to bad prices: $p \to \bar{p} \in \Delta^b$, $p \to \bar{p}' \in \Delta^b$, and the corresponding budget sets converge to different planes: $lim_{p \to \bar{p}} L(p) = L$, $lim_{p' \to \bar{p}'} L(p) = L'$ and still $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$. Then, we should choose $\beta$ so that the condition $(*)$ is satisfied. It is, however, an open question as to whether the decomposition taking care of $(*)$ is generally possible.

We next show how this difficulty is evaded in the GEI economy satisfying the rank condition. A key is the following lemma.

**Lemma 2.** Let $L(p)$ be the budget set of the GEI model where the dimension of $L(p)$ is $k$ or $k-1$ for any $p \in \Delta$. If $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$ for $L, L' \in G_{++}$, $L \neq L'$ and $\mathcal{A} \in \Omega$, then $L = lim_{p \to \bar{p}} L(p)$ and $L' = lim_{p' \to \bar{p}'} L(p)$ for some $\bar{p}, \bar{p}' \in \Delta^b$, $L(\bar{p}) = L(\bar{p}')$, and $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A}) \in L(\bar{p}) = L(\bar{p}')$.

**Proof.** We suppose $L, L \in G_{++}$, $L \neq L'$, and $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$ for an $\mathcal{A} \in \Omega$ and consider when this situation could happen.

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[10] In Momi (2003a) a more general definition of $\mathcal{A}$ is given.

Case 1: If $L$ and $L'$ are elements of $L(\Delta^g)$, then $proj_L(\mathcal{A}) \neq proj_{L'}(\mathcal{A})$ as discussed earlier. (See Momi (2003a) or Bottazzi and Hens (1996) for the proof).

Case 2: Let $L \in L(\Delta^g)$ and $L' \in \overline{L(\Delta^g)} \setminus L(\Delta^g)$. That is, there exists $\bar{p} \in \Delta^g$ such that $L = L(\bar{p})$ and a sequence $p' \to \bar{p}' \in \Delta^b$ such that $L' = lim_{p' \to \bar{p}'} L(p')$.

It is clear that $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$ means $b(\mathcal{A}, L) - a(\mathcal{A}, L) = b(\mathcal{A}, L') - a(\mathcal{A}, L')$. $b(\mathcal{A}, L) - a(\mathcal{A}, L)$, which is in $L^{\perp}$, should be written as

$$b(\mathcal{A}, L) - a(\mathcal{A}, L) = \begin{bmatrix} t\bar{p}_0 \\ (t+\lambda_1)\bar{p}_1 \\ \vdots \\ (t+\lambda_S)\bar{p}_S \end{bmatrix}$$

where $t \in R$ and $\lambda = (\lambda_1, \cdots, \lambda_S)$ satisfies $\lambda A(\bar{p}) = 0$. (See Bottazzi and Hens (1996) for the proof). Similarly, $b(\mathcal{A}, L') - a(\mathcal{A}, L')$, which is in $L'^{\perp}$, should be written as

$$b(\mathcal{A}, L') - a(\mathcal{A}, L') = \begin{bmatrix} t'\bar{p}'_0 \\ (t'+\lambda'_1)\bar{p}'_1 \\ \vdots \\ (t'+\lambda'_S)\bar{p}'_S \end{bmatrix}$$

where $t' \in R$ and $\lambda' = (\lambda'_1, \cdots, \lambda'_S)$ satisfies $\lambda' A(\bar{p}') = 0$. Thus $\bar{p}_s = \alpha_s \bar{p}'_s$ holds with some $\alpha_s \in R$, $s = 0, \ldots, S$. Then $A(\bar{p})$ and $A(\bar{p}')$ have the same rank, hence $L(\bar{p})$ and $L(\bar{p}')$ have the same dimension. A contradiction.

Case 3: Let both $L$ and $L'$ be elements of $\overline{L(\Delta^g)} \setminus L(\Delta^g)$. There exist sequences $p \to \bar{p} \in \Delta^b$ and $p' \to \bar{p}' \in \Delta^b$ such that $L = lim_{p \to \bar{p}} L(p)$ and $L' = lim_{p' \to \bar{p}'} L(p')$. If $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$, then $b(\mathcal{A}, L) - a(\mathcal{A}, L) = b(\mathcal{A}, L') - a(\mathcal{A}, L')$ and, as with the discussion as in the previous paragraph, $\bar{p}_s = \alpha_s \bar{p}'_s$ for each $s$. It is easy to observe that then $L(\bar{p}) = L(\bar{p}')$, and $L(\bar{p}) \subset L$ and $L(\bar{p}') \subset L'$. Because of the rank condition, the dimension of $L(\bar{p})$ is $k-1$. Therefore, if $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$ is not on $L(\bar{p}) = L(\bar{p}')$, then both $L$ and $L'$ should be spanned by $L(\bar{p})$ and $proj_L(\mathcal{A})$, that is, $L = L'$. Thus $proj_L(\mathcal{A}) = proj_{L'}(\mathcal{A})$ should be on $L(\bar{p}) = L(\bar{p}')$ if $L \neq L'$ ■

With this lemma in mind, we prove Proposition 1 by applying the method of Momi (2003a), with slight modification.

Let $\bar{k} = k+1$. For $\mathcal{A}^i \in \Omega$, $i = 1, \ldots, \bar{k}$, and $L \in G^k_{++}(R^M)$, we define $B(\mathcal{A}^1, \ldots, \mathcal{A}^{\bar{k}}, L)$ as the interior of the convex hull of $\bigcup_{i=1}^{\bar{k}} b(\mathcal{A}^i, L)$ on $L$:

$$B(\mathcal{A}^1, \ldots, \mathcal{A}^{\bar{k}}, L) = \left\{ x \in L \;\middle|\; \begin{array}{l} x = \sum_{i=1}^{\bar{k}} r^i b(\mathcal{A}^i, L) \\ \text{where } r^i > 0 \text{ and } \sum_{i=1}^{\bar{k}} r^i = 1 \end{array} \right\}.$$

**Lemma 3.** Let the dimension of $L(p)$ is $k$ or $k-1$ for $p \in \Delta$. Let $G \subset G_{++}$ be a compact set. Then, there exist open connected subsets $B_j \subset G^k_{++}(R^M)$ and $(\mathcal{A}^{j1}, \ldots, \mathcal{A}^{j\bar{k}}) \in \Omega \times \cdots \times \Omega$, $j = 1, \ldots, m$ satisfying

(i) $G \subset \bigcup_{j=1}^{m} B_j$,

(ii) $0 \in B(\mathcal{A}^{j1}, \ldots, \mathcal{A}^{j\bar{k}}, L)$ for $L \in \overline{B_j}$, $j = 1, \ldots, m$

(iii) $proj_L(\mathcal{A}^{ji}) \neq proj_{L'}(\mathcal{A}^{ji})$ for $L, L' \in \overline{B_j}$ such that $L \neq L'$, $i = 1, \ldots, \bar{k}$, $j = 1, \ldots, m$.

(iv) $proj_L(\mathcal{A}^{ji}) \notin L(\bar{p})$ for $L \in \overline{B_j}$ such that $L = lim_{p \to \bar{p} \in \Delta^b} L(p)$, $i = 1, \ldots, \bar{k}$, $j = 1, \ldots, m$.

**Proof.** The proof is essentially same as the proof of Lemma 1 of Momi (2003a). For each $L \in L(\Delta^g)$, we take $\mathcal{A}^{L1}, \ldots, \mathcal{A}^{L\bar{k}} \in \Omega \times \ldots \times \Omega$ so that $0 \in B(\mathcal{A}^{L1}, \ldots, \mathcal{A}^{L\bar{k}}, L)$ and that $B(\mathcal{A}^{L1}, \ldots, \mathcal{A}^{L\bar{k}}, L)$ is relatively open in $L$. One additional modification is that for $L \in G \setminus L(\Delta^g)$, that is, for $L = lim_{p \to \bar{p} \in \Delta^b} L(p)$, we take $\mathcal{A}^{L1}, \ldots, \mathcal{A}^{L\bar{k}} \in \Omega \times \ldots \times \Omega$ so that $proj_L(\mathcal{A}^{Li})$, $i = 1, \ldots, \bar{k}$ are not on $L(\bar{p})$ in addition to the above conditions. How to construct these $(\mathcal{A}^{L1}, \ldots, \mathcal{A}^{L\bar{k}})$ for each $L \in G$ is shown in Momi (2003a).

From the continuity of the function $L' \to proj_{L'}(\mathcal{A}^{Li})$, $0 \in B(\mathcal{A}^{L1}, \ldots, \mathcal{A}^{L\bar{k}}, L')$ and $proj_{L'}(\mathcal{A}^{Li}) \neq proj_{L''}(\mathcal{A}^{Li})$ are satisfied for $L'$ and $L''$ in a small neighborhood $B_L$ of $L$ and $proj_{L'}(\mathcal{A}^{Li}) \notin L(\bar{p})$ is also satisfied for $L' \in B_L$ such that $L' = lim_{p \to \bar{p} \in \Delta^b} L(p)$. We let $B_L$ be sufficiently small so that these are satisfied for $L', L'' \in \overline{B_L}$.

Since $\{B_L\}_{L \in G}$ is an open cover of a compact set $G$, we can choose a finite cover, that is, we can choose $L(1), \ldots, L(m)$ so that $G \subset \bigcup_{j=1}^{m} B_{L(j)}$. Rewriting $L(j)$ as $j$, we obtain the statement of the lemma. ■

**Lemma 4.** Let $z : G \to R^M$ be a continuous function such that $z(L) \in L$. Then there exist positive-valued continuous functions $\beta^{ji} : G^k_{++}(R^N) \to R_{++}$, $j = 1, \ldots, m$, $i = 1, \ldots, \bar{k}$, so that, for $L \in G$,

$$z(L) = \sum_{j,i} \beta^{ji}(L) b(\mathcal{A}^{ji}, L).$$

and $\beta^{ji}$ is constant on $G^k_{++}(R^M) \setminus B_j$

**Proof.** The construction of the $\beta^{ji}$ is the same as that in Momi (2003a). Pick any $\bar{\epsilon} > 0$, and define a continuous function $\psi : G^k_{++}(R^M) \to R^M$ by $\psi(L) = \bar{\epsilon} \sum_{j=1}^{m} \sum_{i=1}^{\bar{k}} b(\mathcal{A}^{ji}, L)$. Because of the condition (ii) in Lemma 3, the function $z(L) - \psi(L)$ can be decomposed as

$$z(L) - \psi(L) = \sum_{i=1}^{\bar{k}} \tilde{\beta}^{ji}(L) b(\mathcal{A}^{ji}, L), \quad \text{for } L \in \overline{B_j} \bigcap G. \tag{1}$$

where $\tilde{\beta}^{ji}$ is a positive-valued continuous function defined on $\overline{B_j} \bigcap G$. We extend $\tilde{\beta}^{ji}$ to a positive-valued continuous function defined on $G^k_{++}(R^M)$. We abuse the notation and write this function $\tilde{\beta}^{ji}$.

Since $\{B_j\}_{j=1}^m$ is an open cover of $G$, we take a smooth partition of unity $\{\varphi_j\}_{j=1}^m$ on $G$ subordinate to $\{B_j\}_{j=1}^m$. That is, each $\varphi_j : \bigcup_j B_j \to R$ is smooth; $0 \le \varphi_j(L) \le 1$; $support(\varphi_j) \subset B_j$; $\sum_{j=1}^m \varphi_j(L) \equiv 1$ for $L \in G$. We then smoothly extend each $\varphi_j$ defined on $\bigcup_j B_j$ to a function defined on $G^k_{++}(R^n)$, letting $\varphi_j(L) = 0$ for $L \notin \bigcup_j B_j$. We use the same notation and write this function $\varphi_j$.

Then, for $L \in G$,

$$\begin{aligned} z(L) - \psi(L) &= \sum_{j=1}^m \varphi_j(L)(z(L) - \psi(L)) \\ &= \sum_{j=1}^m \varphi_j(L) \sum_{i=1}^{\bar{k}} \tilde{\beta}^{ji}(L) b(\mathcal{A}^{ji}, L) \\ &= \sum_{j,i} \varphi_j(L) \tilde{\beta}^{ji}(L) b(\mathcal{A}^{ji}, L), \end{aligned}$$

where in the second equality we used the fact that $\sum_{i=1}^{\bar{k}} \tilde{\beta}^{ji}(L) b(\mathcal{A}^{ji}, L) \neq z(L) - \psi(L) \Rightarrow \varphi_j(L) = 0$. Actually, if the left-hand side of this arrow is satisfied, then $L \notin \overline{B_j} \bigcap G$ from (1). Then $L \in G \setminus \overline{B_j}$ for $L$ in $G$, and hence $\varphi_j(L) = 0$.

From the definition of $\psi(L)$, $z(L) = \sum_{j,i} (\varphi_j(L) \tilde{\beta}^{ji}(L) + \bar{\epsilon}) b(\mathcal{A}^{ji}, L)$. Define $\beta^{ji}(L) = \varphi_j(L) \tilde{\beta}^{ji}(L) + \bar{\epsilon}$. It is clear that this $\beta^{ji}$ satisfies the requirement. Especially, $\beta^{ji}(L) = \bar{\epsilon}$ constant, for $L \notin B_j$, because $\varphi_j(L) = 0$ for $L \notin B_j$. ■

Through Lemmas 3 and 4, we show how to decompose a candidate function $L \to z(L)$ to $\beta^{ji}(L) proj_L(\mathcal{A}^{ji})$, $j = 1, \dots, m$, $i = 1, \dots, \bar{k}$. Note that $\beta^{ji}(L) = \bar{\epsilon}$ constant for $L \notin B_j$, $proj_L(\mathcal{A}^{ji}) \neq proj_{L'}(\mathcal{A}^{ji})$ for $L$ and $L'$ in $B_j$, and $proj_L(\mathcal{A}^{ji}) \neq proj_{L'}(\mathcal{A}^{ji})$ for $L \in B_j$ and $L' \notin B_j$ because of Lemma 2 and (iv) in Lemma 3. Each pair of $\beta^{ji}$ and $\mathcal{A}^{ji}$, then, satisfies the condition $(*)$ in Lemma 1. Then, by applying the lemma, we can construct a consumer $(\precsim^{ji}, \omega^{ji})$ who realizes $\beta^{ji}(L) proj_L(\mathcal{A}^{ji})$ as his/her excess demand function for $L \in G_{++}$. This ends the proof of Proposition 1 and Corollary.

## 4.3 Characterization of equilibrium set in GEI model

We refine Proposition 1 (Corollary) to the GEI model version of Mas-Colell's theorem. We define some notations. To measure the distance between $L \in G_{++}$ and the boundary $\partial G^k_{++}(R^M)$, we fix a set $\mathcal{A}^0 \in \Omega$. Since $d(\mathcal{A}^0, L) = \| b(\mathcal{A}^0, L) - a(\mathcal{A}^0, L) \| \to 0$ as $L \to$

$\bar{L} \in \partial G^k_{++}(R^M)$, $d(\mathcal{A}^0, L)$ works as an indicator to measure the distance of $L$ from the boundary. For $\epsilon > 0$, we let

$$G_\epsilon = \{L \in G_{++} | d(\mathcal{A}^0, L) > \epsilon\}.$$

For any function $z : G_{++} \to R^M$, we define

$$E_z = \{L \in G_{++} | z(L) = 0\}.$$

That is, $E_z$ denotes the set of equilibrium budget sets when $z$ is the aggregate excess demand function defined on $G_{++}$.

**Proposition 2.** Let the asset structure satisfy the rank condition. Let $z : G_{++} \to R^M$ be a continuous function satisfying (W')(BB')(BC'). For sufficiently small $\epsilon$, there exists an economy $\{\precsim^i, \omega^i\}_i$ of a finite number of consumers whose aggregate excess demand function $z^*$ satisfies that $z^*|G_\epsilon = z|G_\epsilon$, $E_{z^*} = E_z \subset G_\epsilon$, and $\sum_{l=1}^M z^*_l(L) > 0$ for $L \in G_{++} \setminus G_\epsilon$.

**Proof.** The proof is essentially same as that of Mas-Colell (1977). Let $\epsilon$ be sufficiently small so that $\sum_l z_l(L) > 0$ and $\sum_l b_l(\mathcal{A}^0, L) > 0$ for $L \in G_{++} \setminus G_\epsilon$.

By Lemma 3, there exist open connected subsets $B_j \subset G^k_{++}(R^M)$ and $(\mathcal{A}^{j1}, \ldots, \mathcal{A}^{j\bar{k}}) \in \Omega \times \cdots \times \Omega$, $j = 1, \ldots, m$ such that (i) $\overline{G_{\frac{\epsilon}{2}}} \subset \bigcup_{j=1}^m B_j$ and (ii)–(iv) of Lemma 3. In other words, any function on $\overline{G_{\frac{\epsilon}{2}}}$ can be decomposed by using these $\mathcal{A}^{ji}$'s as in Lemma 4.

We pick any $\bar{\epsilon} > 0$. For $\beta^0 > 0$, we let $F^{\beta^0} : G_{++} \to R^M$ be a function defined by

$$F^{\beta^0}(L) = \beta^0 proj_L(\mathcal{A}^0) + \bar{\epsilon} \sum_{j,i} proj_L(\mathcal{A}^{ji}),$$

and fix $\beta^0$ so that $\sum_{l=1}^M F^{\beta^0}_l(L) > 0$ for any $L \in G_{++} \setminus G_\epsilon$. From the definition of $\epsilon$, the definition of $\mathcal{A}^0$ and the boundary behavior of $L \mapsto proj_L(\mathcal{A}^{ji})$, this condition is satisfied for suitably large $\beta_0$.

As shown in Lemma 4, we decompose the function $L \to z(L) - \beta^0 proj_L(\mathcal{A}^0)$ on $\overline{G_{\frac{\epsilon}{2}}}$ using $\mathcal{A}^{ji}$s defined above so that $\beta^{ji}(L) = \bar{\epsilon}$ for $L \notin B_j$. That is, we determine positive-valued continuous functions $\beta^{ji} : G^k_{++}(R^n) \to R_{++}$ such that

$$z(L) = \beta^0 proj_L(\mathcal{A}^0) + \sum_{j,i} \beta^{ji}(L) proj_L(\mathcal{A}^{ji}) \text{ for } L \in \overline{G_{\frac{\epsilon}{2}}},$$

and $\beta^{ji}(L) = \bar{\epsilon}$ for $L \notin B_j$.

Let $\eta : G_{++} \to [0, 1]$ be a smooth function such that $\eta(L) = 0$ when $L \in G_\epsilon$ and $\eta(L) = 1$ when $L \in G_{++} \setminus G_{\frac{\epsilon}{2}}$. Define $z^0(L) = \beta^0 proj_L(\mathcal{A}^0)$ and

$$z^{ji}(L) = \{(1 - \eta(L))\beta^{ji}(L) + \eta(L)\bar{\epsilon}\} proj_L(\mathcal{A}^{ji}).$$

Note that the coefficient of the above equation $(1-\eta(L))\beta^{ji}(L)+\eta(L)\bar{\epsilon}$ is still $\bar{\epsilon}$ constant for $L \notin B_j$ Therefore each pair of $\beta^{ji}$ and $\mathcal{A}^{ji}$ satisfies the condition $(*)$ in Lemma 1.

By applying Lemma 1, we can construct $(\precsim^0, \omega^0)$ and $(\precsim^{ji}, \omega^{ji})$, $j = 1, \ldots, m$, $i = 1, \ldots, \bar{k}$, which respectively generate $z^0$ and $z^{ji}$ as their excess demands. The aggregate excess demand function of the economy $\{(\precsim^0, \omega^0), \{(\precsim^{ji}, \omega^{ji})\}_{ji}\}$ is $z^* = z^0 + \sum_{ji} z^{ji}$.

We finally check that this economy and the aggregate excess demand function $z^*$ satisfies the requirements. When $L \in G_\epsilon$, $\eta(L) = 0$. Then $z^*(L) = \beta^0 proj_L(\mathcal{A}^0) + \sum_{ji} \beta^{ji}(L) proj_L(\mathcal{A}^{ji}) = z(L)$. Therefore $z^*|G_\epsilon = z|G_\epsilon$. When $L \in G_{++} \setminus G_{\frac{\epsilon}{2}}$, $\eta(L) = 1$. Then, $z^*(L) = \beta^0 proj_L(\mathcal{A}^0) + \bar{\epsilon} \sum_{j,i} proj_L(\mathcal{A}^{ji})$, hence $\sum_{l=1}^M z^*_l > 0$ because of our definition of $\beta^0$. When $L \in G_{\frac{\epsilon}{2}} \setminus G_\epsilon$, $z^*(L) = (1-\eta(L))z(L) + \eta(L)(\beta^0 proj_L(\mathcal{A}^0) + \bar{\epsilon} \sum_{j,i} proj_L(\mathcal{A}^{ji}))$, hence $\sum_{l=1}^M z^*_l > 0$ because of the definitions of $\beta^0$ and $\epsilon$. Therefore no equilibrium budget set exists in $G \setminus G_\epsilon$ and $E_z = E_{z^*} \subset G_\epsilon$. ■

## 5 Homotopy, index, and index change rule in GEI model

In this section, we define the homotopy and the index of the GEI model given in Section 3 and review the properties of the homotopy path and the index change rule. This section therefore parallels Section 2.1 for the Arrow–Debreu economy.

Our definition of the homotopy of the GEI model is the same as that in Brown et al. (1996). We assume the first agent to be unconstrained, maximizing his/her utility subject to $pz = 0$, hence his/her excess demand function is $z^1(p) = \{x - \omega^1 \in R^M | p(x-\omega^1) = 0 \text{ and } x' \precsim^i x \text{ for any } x' \in R^M_+ \text{ satisfying } p(x'-\omega^1) = 0\}$. We let $p^u$ denote the supporting price of his/her indifference surface at his/her initial endowment $\omega^1$. Other consumers are constrained, maximizing their utility subject to the budget set $L(p)$, hence their excess demand functions are $z^i(p) = \{x - \omega^i \in R^M | (x-\omega^i) \in L(p) \text{ and } x' \precsim^i x \text{ for any } x' \in R^M_+ \text{ satisfying } (x'-\omega^i) \in L(p)\}$, $i \geq 2$. We let $Z^c = \sum_{i \neq 1} z^i$ denote the aggregate excess demand function of the constrained consumers and let $Z = z^1 + Z^c = \sum_i z^i$ denote the aggregate excess demand of the economy. It is well known, as shown in Duffie and Shafer (1985), that $\hat{Z}(\bar{p}) = 0$, where "ˆ" is the symbol to drop the last element, implies that $\bar{p}$ is an equilibrium price. It is sometimes convenient to recognize excess demand functions as functions of the budget set. For a linear subspace $L$ of $R^M$ such that $L \bigcap R_+ = \{0\}$, we define $\tilde{z}^i(L) = \{x - \omega^i \in R^M | (x-\omega^i) \in L \text{ and } x' \precsim^i x \text{ for any } x' \in R^M_+ \text{ satisfying } (x'-\omega^i) \in L\}$ for any $i$ and define $\tilde{Z}^c = \sum_{i \neq 1} \tilde{z}^i$. It is clear that $\tilde{z}^i(L(p)) = z^i(p)$ for $i \geq 2$ while $\tilde{z}^1(L(p)) \neq z^1(p)$. Note that if $z^1(p) \in L(p)$ then $\tilde{z}^1(L(p)) = z^1(p)$ because $z^1(p) + \omega^i$

achieves the maximum utility in $L(p) + \omega^i$.

We define homotopy $H : \Delta^g \times [0,1] \to R^M$ as $H(p,t) = z^1(p) + tZ^c(p)$. Note that, though $z^1(p)+tZ^c(p)$ is continuous on $\Delta^g \times [0,1]$, it is not continuous at $\Delta^b \times [0,1]$, hence we have defined $H$ on $\Delta^g \times [0,1]$. It is sometimes convenient to work with $\overline{H^{-1}(0)}$, the closure of $H^{-1}(0)$ in $\Delta \times [0,1]$

We list properties of $H^{-1}(0)$ satisfied for almost all initial endowments $\omega$ and asset structure $A$ when each of $\precsim^i$ is represented by a smooth utility function. The following properties, (I1)–(I5), are well known, and are found in Brown et al. (1996) and Momi (2003b).

(I1) $\overline{H^{-1}(0)}$ is a one-dimensional smooth compact submanifold of $\Delta \times [0,1]$;

(I2) $\overline{H^{-1}(0)}$ is bounded away from the boundary $\partial\Delta \times [0,1]$;

(I3) $\overline{H^{-1}(0)}$ is not tangential to the boundary $\Delta \times \{0,1\}$;

(I4) $p^u \in \Delta^g$ and $p^u$ is the unique solution of $H(\cdot,0) = 0$;

(I5) $\overline{H^{-1}(0)}$ intersects $\Delta^b \times [0,1]$ a finite times.

If $H(p,t) = 0$, then $z^1(p) = -tZ^c(p)$ and $Z^c(p)$ is on $L(p)$. Therefore,

(I6) for $(p,t) \in H^{-1}(0)$, $z^1(p) \in L(p)$.

Let $(p,t)$ and $(p',t')$ be in $H^{-1}(0)$ and $L(p) = L(p')$. Since $L(p) = L(p')$ means $Z^c(p) = Z^c(p')$, we have $z^1(p) = -tZ^c(p)$ and $z^1(p') = -t'Z^c(p)$. That is, both $z^1(p)$ and $z^1(p')$ are on a line spanned by the vector $Z^c(p)$. If $Z^c(p) = 0$, then $z^1(p) = 0$, hence $p = p^u$. However $Z^c(p^u) = 0$ is not robust to perturbations of $\{\omega_i\}_{i\neq 1}$. If $Z^c(p) \neq 0$, then $p = p'$ and $t = t'$ should hold because of the strict convexity of the first consumer's preference ordering. That is,

(I7) if $(p,t)$ and $(p',t')$ are in $H^{-1}(0)$ and $L(p) = L(p')$, then $p = p'$ and $t = t'$.

Suppose the above properties are satisfied. We can then parameterize each connected one-dimensional manifold in $H^{-1}(0)$ as $(p(s), t(s)) \in \Delta^g \times [0,1]$, where $s$ is a parameter. Without loss of generality, we assume $(\frac{\partial p}{\partial s}, \frac{\partial t}{\partial s}) \neq (0,0)$ for the parameterization. By taking the derivative of $z^1(p(s)) + t(s)Z^c(p(s)) = z^1(p(s)) + t(s)\tilde{Z}^c(L(p(s))) = 0$, we have

$$\left[\frac{\partial z^1}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] + t\left[\frac{\partial \tilde{Z}^c}{\partial L}\right]\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] + \tilde{Z}^c\frac{\partial t}{\partial s} = 0.^{11}$$

Suppose $\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] = 0$. If $\left[\frac{\partial p}{\partial s}\right] = 0$, then $\frac{\partial t}{\partial s} \neq 0$ from our parameterization. If $\left[\frac{\partial p}{\partial s}\right] \neq 0$, then $\left[\frac{\partial z^1}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] \neq 0$, hence again $\frac{\partial t}{\partial s} \neq 0$. We have thus proved that if $\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] = 0$, then $\frac{\partial t}{\partial s} \neq 0$.

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[11] $L$ is identified with its local coordinate. We will see the formal definition later.

As in (I6), $z^1(p(s)) \in L(p(s))$, hence $\tilde{z}^1(L(p(s))) = z^1(p(s))$. Then, by taking the derivative of $\tilde{z}(L(p(s))) + t(s)\tilde{Z}^c(L(p(s))) = 0$ with respect to $s$, we obtain

$$\left[\frac{\partial \tilde{z}^1}{\partial L} + t\frac{\partial \tilde{Z}^c}{\partial L}\right]\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] + \tilde{Z}^c\frac{\partial t}{\partial s} = 0.$$

If $\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] = 0$, then $\tilde{Z}^c(L(p(s))) = 0$ because $\frac{\partial t}{\partial s} \neq 0$ as shown above, hence $\tilde{z}^1(L(p(s)) = z^1(p(s)) = 0$ and $p(s) = p^u$. However $\tilde{Z}^c(L(p^u)) = Z^c(p^u) = 0$ is not robust to perturbations of $\{\omega_i\}_{i\neq 1}$. Therefore $\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] \neq 0$, that is

(I8) the budget set $L(p)$ changes according to infinitesimal movements of $p$ along the path.

It is known that $\left[\frac{\partial p}{\partial s}\right] \notin L(p)^{\perp}$ assures $\left[\frac{\partial L}{\partial p}\right]\left[\frac{\partial p}{\partial s}\right] \neq 0$. (See Claim 2 in Appendix for the proof.)

Typically $H^{-1}(0)$ is drawn as depicted in Figure 2. We denote the one-dimensional connected manifold as path and define path in $H^{-1}(0)$ or in $\overline{H^{-1}(0)}$ as homotopy path. We denote each point $(p, t)$ in $\overline{H^{-1}(0)}$ as a good point if $p$ is a good price, that is, if $(p, t) \in H^{-1}(0)$, and a bad point if $p$ is a bad price, that is, if $(p, t) \in \overline{H^{-1}(0)} \setminus H^{-1}(0)$.

We define the index at an equilibrium price $\bar{p} \in \hat{Z}^{-1}(0)$ by

$$index\hat{Z}(\bar{p}) = (-1)^{M-1} sign\left(det\left[\frac{\partial \hat{Z}}{\partial \hat{p}}(\bar{p})\right]\right),$$

and the index of the economy is defined by

$$\sum_{\bar{p}\in\hat{Z}^{-1}(0)} index\hat{Z}(\bar{p}).$$

This definition is equivalent to that found in Momi (2003b) regardless of the different price normalizations and is a natural extension of the index of the Arrow–Debreu economy. (See Momi (2003b) for the details.) The index of an economy is, therefore, well defined only when $\hat{Z}^{-1}(0)$ is a finite set and the derivative of $\hat{Z}$ is well defined at every $\bar{p} \in \hat{Z}^{-1}(0)$. What is known about the index of the GEI economy is the next theorem.

**Theorem 4.** [Momi (2003b)] *Let each $\precsim^i$ be represented by a smooth utility function and let $S-J$ be even. Then the index of the GEI economy is $+1$ for almost all endowments $\omega$ and asset structure $A$.*

In the rest of this section, we review how this theorem is proved. We define index of $H$ at $(\bar{p}, \bar{t}) \in H^{-1}(0)$ as $index\hat{H}(\bar{p}, \bar{t}) = (-1)^{M-1} sign\left(det\left[\frac{\partial \hat{H}}{\partial \hat{p}}(\bar{p}, \bar{t})\right]\right)$. On a path in $H^{-1}(0)$, the index rule the same as that in the Arrow–Debreu model holds. It is well known that, when each of $\precsim^i$ is represented by a smooth utility function, for almost all initial endowments $\omega$ and payoff structure $A$, the following, (R1) and (R2), hold.

(R1)The index of $H$ at $(p^u, 0) \in H^{-1}(0)$ is $+1$.

(R2) Regularity of $H$: the $(M-1) \times M$ matrix $\left[\frac{\partial \hat{H}}{\partial \hat{p}}, \frac{\partial \hat{H}}{\partial t}\right]$ has rank $M-1$ at all $(p, t) \in H^{-1}(0)$,

and (R2) gives the index rule on $H^{-1}(0)$ that determines the relation of indices at two different points on a path in $H^{-1}(0)$

(R2') When we follow a homotopy path in $H^{-1}(0)$, the index of $H$ is unchanged as long as we move in the same direction with respect to $t$ and is changed when the direction is changed.

The problem is that $H^{-1}(0)$ does not necessarily give a connected path that has both end points on the boundary $\Delta \times \{0, 1\}$. See Figure 2. The rule (R2') determines the relation between index of $H$ at $(p', t')$ and index of $H$ at $(p'', t'')$. However, since $H^{-1}(0)$ is not connected, this rule reveals nothing about the index of $H$ at $(p''', t''')$. What Momi (2003b) proved is the index change rule that determines the relation between indices at $(p'', t'')$ and $(p''', t''')$. That is, it tells how the index of $H$ changes when we follow a path in $\overline{H^{-1}(0)}$ and passes through a bad point where $H^{-1}(0)$ is not connected. The index change rule is summarized as follows.

Let $(\bar{p}, \bar{t}) \in \overline{H^{-1}(0)} \setminus H^{-1}(0)$ be a bad point. Though the payoff matrix

$$A(p) = \begin{bmatrix} p_1 A_1^1 & \dots & p_1 A_1^J \\ \vdots & & \vdots \\ p_S A_S^1 & \dots & p_S A_S^J \end{bmatrix}$$

drops its rank at this point, it is , of course, of full column rank at any $(p, t) \in H^{-1}(0)$ in a sufficiently small neighborhood of $(\bar{p}, \bar{t})$ and we can choose $J$ rows of $A(p)$ so that the $J \times J$ matrix that consists of these rows is full rank at $(p, t) \in H^{-1}(0)$ in a neighborhood of $(\bar{p}, \bar{t})$. We write this $J \times J$ matrix as $Q_{(\bar{p}, \bar{t})}(p)$. Though this matrix $Q_{\bar{p}, \bar{t})}(p)$ is defined at each bad point $(\bar{p}, \bar{t})$, we often write $Q(p)$ for simplicity when there is no confusion.

**Lemma 5.** [Momi (2003b)] When we follow a path in $\overline{H^{-1}(0)}$, the index change occurs at a bad point $(\bar{p}, \bar{t}) \in \overline{H^{-1}(0)} \setminus H^{-1}(0)$ if $(det[Q_{(\bar{p}, \bar{t})}(p)])^{S-J}$ changes its signs before and after the bad point.

We call (R1)(R2') and this lemma as the index change rule for the GEI model. When a GEI economy is given, we can compute its index by applying this rule.

If $S-J$ is even, in a neighborhood of each bad point, $(det[Q(p)])^{S-J}$ does not change its signs regardless of the sign of $det[Q(p)]$, hence the index changes at bad points do not occur. That is, we can ignore bad points and consider the paths in $\overline{H^{-1}(0)}$ as the standard homotopy paths and prove the index theorem (Theorem 4) by applying the index rule (R1)(R2') as in the case of the Arrow–Debreu economy.

On the other hand, if $S-J$ is odd, the index change at a bad point occurs when $det[Q(p)]$ changes its signs before and after the bad point. This suggests the possibility that an economy has an index that is not +1. Whether $det[Q(p)]$ changes its signs at a bad point depends only on how the homotopy path passes through the bad point. If we have a free hand in deciding the paths and index number (+1 or -1) at one end point of each path, we can induce any odd number as the sum of indices at the end points of the paths on the boundary $\Delta \times \{1\}$. The problem we have to consider is, of course, whether there is an economy $\{\precsim^i, \omega^i\}_i$ that actually realizes the drawn paths and the given index number at one end point of each path as its homotopy paths and the index of the homotopy.

## 6 Theorem and proof

We are now ready to prove our main result: for any odd number, there is an economy that has the odd number as its index. Recall Theorem 1 for the Arrow–Debreu economy and the sketch of its proof in Section 2. Our discussion parallels these.

For an index number not +1 to be realized, $S-J$ should be an odd number and $\Delta^b$ should not be an empty set. Since we apply Proposition 2 with the economy's construction, we assume that our asset structure satisfies the rank condition: the dimension of the budget set $L(p)$ is $k$ or $k-1$, that is the rank of $A(p)$ is $J$ or $J-1$, for any $p \in \Delta$. We fix an asset structure $A$ satisfying these conditions. (For example, let $S=3$, $J=2$, $N=2$ and $A^j_{sn} > 0$.)

**Theorem 5.** Let $S-J$ be an odd number and $A$ be an asset structure satisfying the rank condition and $\Delta^b \neq \emptyset$. For any odd number, there exists an economy $\{\precsim^i, \omega^i\}_i$ of a finite number of consumers whose index is the odd number.

The remainder of this paper is devoted to the proof of this theorem. Let $\alpha$ be an odd number. We construct an economy whose index is $\alpha$. The proof proceeds parallel to the sketch for the Arrow–Debreu economy in Section 2. Roughly speaking, we will first fix the first consumer. Second, we will draw a picture of the paths that induces index $\alpha$ when the index change rule is applied. Third, we will construct a candidate aggregate excess demand function of constrained consumers so that the homotopy paths coincide with the drawn paths. Finally, we will apply Proposition 2 and construct the constrained consumers whose aggregate excess demand function coincides with candidate one in the previous step.

## 6.1 Construction of the first consumer

We determine the unconstrained first consumer $(\precsim^1, \omega^1)$ so that the supporting price $p^u$ of his/her indifference surface at his/her endowment $\omega^1$ is a good price: $p^u \in \Delta^g$ and his/her excess demand function $z^1(p)$ is smooth.

## 6.2 Construction of paths

We represent candidate homotopy paths (candidate paths of $\overline{H^{-1}(0)}$) as $(p^j(s), t^j(s))$, $j = 1, \ldots, r$ parameterized by $s \in [0,1]$. Without loss of generality, we assume that $(\frac{\partial p^j}{\partial s}(s), \frac{\partial t^j}{\partial s}(s)) \neq (0,0)$ at any $s$. We first list the conditions we require for the paths.

(i) $p^j : [0,1] \to \Delta$ and $t^j : [0,1] \to [0,1]$ are smooth functions;

(ii) $p^j(s)$ are bounded away from the boundary $\partial\Delta$;

(iii) $\frac{\partial t^j}{\partial s}(0) \neq 0$ and $\frac{\partial t^j}{\partial s}(1) \neq 0$ ;

These requirements, of course, correspond to (I1)–(I3) in the previous section. For simplicity, assume that the first path whose one end point is $(p^u, 0)$ does not have a bad point, while each of the others has one bad point where index change occurs. That is,

(iv) $(p^1(0), t^1(0)) = (p^u, 0), t^1(1) = 1$, and $t^j(0) = t^j(1) = 1$, $j \geq 2$;

(v) $p^1(s) \in \Delta^g$ for all $s \in [0,1]$;

(vi) for each $j \geq 2$, $p^j(s) \in \Delta^g$ for all $s$ except one parameter value $\bar{s}^j \in (0,1)$: $p^j(\bar{s}^j) \in \Delta^b$ and the $det[Q(p)]$ changes its signs at the bad point $(p^j(\bar{s}^j), t^j(\bar{s}^j))$.

Formally we define $\bar{s}^1 = \emptyset$ and define $L^j : [0,1] \to G_{++}$ by

$$\begin{cases} L^j(s) = L(p^j(s)) \text{ for } s \neq \bar{s}^j \\ L^j(\bar{s}^j) = lim_{s \to \bar{s}^j} L(p^j(s)) \end{cases} . \tag{2}$$

Note that, when $p^j(s)$ is smooth, $L^j(s)$ is also smooth, while $L(p^j(s))$ is not continuous at $\bar{s}$. We further require the paths to satisfy

(vii) $z^1(p^j(s)) \in L(p^j(s))$ for $s \in [0,1] \setminus \bar{s}^j$;

(viii) $L^{j'}(s') = L^{j''}(s'') \Rightarrow (j', s') = (j'', s'')$;

(ix) $\left[\frac{\partial p^j}{\partial s}(s)\right] \notin L^j(s)$ .

These are required to establish (I6)–(I8) in the previous section.

When paths are fixed, the index possible is determined. In Figure 2, suppose that $det[Q(p)]$ changes its signs at every bad point. Then the possible index is $+1$ or $-3$. While the index of the path with one end point on $\Delta \times \{0\}$ is determined unambiguously, the index of each of the other paths with both end points on $\Delta \times \{1\}$ has two possibilities. When a set of paths is not inconsistent with an odd number in this sense, we say the paths are consistent with the odd number. For example, the paths in the figure are consistent with $+1$ and $-3$. Our requirement is, of course,

(x) $(p^j(s), t^j(s))$, $j = 1, \ldots, r$, are consistent with $\alpha$.

Paths satisfying the above conditions can be constructed as follows. It is evident that we can draw paths $(\tilde{p}^j(s), t^j(s))$, $j = 1, \ldots, r$, satisfying the conditions except (vii).

We write the function (2) corresponding to $\tilde{p}^j(s)$ by $\tilde{L}^j$: $\tilde{L}^j(s) = L(\tilde{p}^j(s))$ and $\tilde{L}^j(\bar{s}^j) = lim_{s \to \bar{s}^j} L(\tilde{p}^j(s))$. We define a new price vector $p^j(s)$ so that $p^j(s)$ is normal to the first consumer's indifference surface at $\tilde{z}^1(\tilde{L}(s)) + \omega^1$. Then both $\tilde{p}^j(s)$ and $p^j(s)$ are normal to $L(\tilde{p}^j(s))$ and this implies $L(p^j(s)) = L(\tilde{p}^j(s))$. (See Claim 1 in Appendix). Hence $z^1(p^j(s)) = \tilde{z}^1(L(\tilde{p}^j(s))) \in L(p^j(s))$ at $s \neq \bar{s}^j$ as desired. It is easy to observe that sign of $det[Q]$ at $p^j(s)$ and that at $\tilde{p}^j(s)$ coincide. Thus $(p^j(s), t^j(s))$, $j = 1, \ldots, r$, satisfy all requirements.

Note that we can draw the first path in an arbitrarily small neighborhood of $p^u \times [0, 1]$, and each of the other paths in an arbitrarily small neighborhood of $\bar{p}^j \times [0, 1]$, where $\bar{p}^j = p^j(\bar{s}^j)$ is the bad price the path $j$ passes through. Therefore, for the purposes of simplicity in later discussion, we assume that the variation of the budget set $L^j(s)$ on each path is sufficiently small:

(xi) $\max_{s', s'' \in [0,1]} \tilde{d}(L^j(s') - L^j(s''))$ is sufficiently small, $j = 1, \ldots, r$,
with respect to a metric $\tilde{d}$ introduced on $G^k_{++}(R^M)$.

## 6.3 Construction of excess demand of constrained consumers

We have determined the first consumer and the paths. In this section, we construct a candidate aggregate excess demand function of constrained consumers. For a function $F : G^k_{++}(R^M) \to R^M$ we define the homotopy $H_F : \Delta^g \times [0, 1] \to R^M$ between $z^1(p)$ and $F(L(p))$ as $H_F(p, t) = z^1(p) + tF(L(p))$ and define the index of $H_F$ analogously to that of $H$. We want to obtain a smooth function $F : G^k_{++}(R^M) \to R^M$ such that

(F1) $F$ satisfies Walras' law: $F(L) \in L$;

(F2) each path $(p^j(s), t^j(s))$, $j = 1, \ldots, r$, is a regular homotopy path of $H_F(p, t) = 0$;

(F3) the sum of the indices of $H_F$ at $(p^1(1), t^1(1))$, $(p^j(0), t^j(0))$ and $(p^j(1), t^j(1))$, $j = 2, \ldots, r$, is $\alpha$.

We first show how we can obtain $F$ satisfying (F1) and (F2). When (F1) is satisfied, the regularity condition (F2) is decomposed into the following two conditions.

$$H_F(p^j(s), t^j(s)) = z^1(p^j(s)) + t^j(s)F(L(p^j(s))) = 0 \text{ for } s \neq \bar{s}^j, \tag{3}$$

$$dH_F : T_{(p^j(s),t^j(s))}(\Delta \times [0,1]) \to p^j(s)^\perp \text{ is surjective }^{12} \text{ for } s \neq \bar{s}^j. \tag{4}$$

It is well known that, since $H_F(p, t) \in p^\perp$, $dH_{F,(p,t)}$ is a map into $T_p\Delta = p^\perp$ at $(p, t)$ satisfying $H_F(p, t) = 0$. Therefore, (4) implies the regularity of the paths. We rephrase these by the constrained excess demand of the first consumer $\tilde{z}^1$, instead of $z^1$.

**Lemma 6.** If $F : G^k_{++}(R^M) \to R^M$ is a function such that $F(L) \in L$ and

$$\tilde{H}_F(p^j(s), t^j(s)) \equiv \tilde{z}^1(L(p^j(s))) + t^j(s)F(L(p^j(s))) = 0 \text{ for } s \neq \bar{s}^j, \tag{5}$$

$$d\tilde{H}_F : T_{(p^j(s),t^j(s))}(\Delta \times [0,1]) \to L(p^j(s)) \text{ is surjective for } s \neq \bar{s}^j, \tag{6}$$

then $F$ satisfies (3) and (4).

**Proof.** See Appendix. ■

For each path $j = 1, \ldots, r$, let $B^j \subset G^k_{++}(R^M)$ be a sufficiently small neighborhood of $\bigcup_s L^j(s)$. We construct $F^j$ defined on $B^j$ satisfying (F1),(5) and (6), and then extend these to $F$ so that $F|B^j = F^j$.

We let $(B^j, \varphi^j)$ be a local coordinate of the Grassmanian $G^k_{++}(R^M)$: $\varphi^j : B^j \to R^{k(M-k)}$ and let $y = (y_1, \ldots, y_{k(M-k)})$ be the coordinate function of $R^{k(M-k)}$. Since $F$ should satisfy Walras' law, only $k$ elements of $(F^j_1, \ldots, F^j_M)$ are independent. Because of (xi) in the previous section, each $B^j$ is sufficiently small, hence, without loss of generality, we assume that, on $B^j$, the last $M - k$ elements of $F$ can be computed from the first $k$

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[12] For a smooth map $f : X \to Y$, $df_x : T_xX \to T_{f(x)}Y$ denotes the differential of $f$ at $x \in X$, where $T_xX$ and $T_{f(x)}Y$ are the tangent spaces of $X$ and $Y$ at $x$ and $f(x)$, respectively. We also write $df : TX \to TY$ when $x$ and $f(x)$ are clear from the context.

elements, $\check{F}^j = (F^j_1, \dots, F^j_k)$. We use the symbol "ˇ" to denotes the first $k$ elements of a vector in $R^M$.

Observe that (5) and (6) are conditions for value and derivative of $F^j$ on the path. Therefore, we first determine a candidate value and a candidate derivative of $F^j$ at $(p^j(s), t^j(s))$ satisfying (5) and (6), and then construct $F^j$ that actually attains them.

The value of $F^j$ at $(p^j(s), t^j(s))$ is determined uniquely from (5). Let $\Psi^j : [0,1] \to R^k$ be a smooth function defined by

$$\Psi^j(s) = -\frac{1}{t^j(s)}\check{\tilde{z}}^1(L^j(s))$$

where $\Psi^1(0) = lim_{s\to 0} - \frac{1}{t^1(s)}\check{\tilde{z}}^1(L^1(s))$.[13] Clearly, the value of $\check{F}^j$ at $L^j(s)$ should be $\Psi^j(s)$, or equivalently the value of $\check{F}^j \circ (\varphi^j)^{-1}$ at $\varphi^j((L^j(s)))$ should be $\Psi^j(s)$.

We next consider a candidate derivative of $\check{F}^j$. By taking the derivative of (5) with respect to $s$, we obtain a necessary condition the derivative of $\check{F}$ has to satisfy: at $s \neq \bar{s}$,

$$\left[\left[\frac{\partial \check{\tilde{z}}^1}{\partial L}(L(p^j(s))) + t^j(s)\frac{\partial \check{F}^j}{\partial L}(L(p^j(s)))\right]\left[\frac{\partial L}{\partial p}(p^j(s))\right], \check{F}^j(L(p^j(s)))\right]\begin{bmatrix}\frac{\partial p^j}{\partial s}(s)\\ \frac{\partial t^j}{\partial s}(s)\end{bmatrix} = 0, \quad (7)$$

where

$$\left[\frac{\partial \check{\tilde{z}}^1}{\partial L}(L(p^j(s)))\right] \equiv \left[\frac{\partial(\check{\tilde{z}}^1 \circ (\varphi^j)^{-1})}{\partial y^j}(\varphi^j(L(p^j(s))))\right],$$

$$\left[\frac{\partial \check{F}^j}{\partial L}(L(p^j(s)))\right] \equiv \left[\frac{\partial(\check{F}^j \circ (\varphi^j)^{-1})}{\partial y^j}(\varphi^j(L(p^j(s))))\right],$$

and

$$\left[\frac{\partial L}{\partial p}(p^j(s))\right] \equiv \left[\frac{\partial(\varphi^j \circ L)}{\partial p}(p^j(s))\right].$$

Note that (6) is satisfied when the column vectors of the $k \times (M+1)$ matrix in (7)

$$\left[\left[\frac{\partial \check{\tilde{z}}^1}{\partial L}(L(p^j(s))) + t^j(s)\frac{\partial \check{F}^j}{\partial L}(L(p^j(s)))\right]\left[\frac{\partial L}{\partial p}(p^j(s))\right], \check{F}(L(p^j(s)))\right] \quad (8)$$

spans $R^k$ at any $s \neq \bar{s}^j$.

We take $k-1$ smooth maps $\alpha^j_2(s), \dots, \alpha^j_k(s)$ so that $\alpha^j_l(s) \in dL(T_{p^j(s)}\Delta)$ for $s \neq \bar{s}^j$, $l = 2, \dots, k$ and $\frac{\partial L^j}{\partial s}(s)$, $\alpha^j_2(s), \dots, \alpha^j_k(s)$ are always independent. (See Claim 3 in Appendix for the existence of such $\alpha^j_l$'s.) We also take $k-1$ smooth maps $\beta^j_2(s), \dots, \beta^j_k(s)$ into $R^k$

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[13] $\frac{\partial t^1}{\partial s}(0) \neq 0$ assures the existence of this limit.

so that $F(L^j(s)), \beta_2^j(s), \dots, \beta_k^j(s)$ are always independent. We then obtain $k \times k(M-k)$ matrix $\Phi^j(s)$ smooth with respect to $s$ satisfying

$$\left[\frac{\partial \check{z}^1}{\partial L}(L^j(s)) + t^j(s)\Phi^j(s)\right]\left[\frac{\partial L^j}{\partial s}(s), \alpha_2^j(s), \dots, \alpha_k^j(s)\right]$$
$$= \left[-\check{F}^j(L^j(s))\frac{\partial t^j}{\partial s}(s), \beta_2^j(s), \dots, \beta_k^j(s)\right].$$

The existence of such an $\Phi^j(s)$ is clear because the number of the element of $\Phi^j(s)$ is $k \times k(M-k)$ while the system consists of $k \times k$ equations. It would be clear that $\Phi^j(s)$ is a candidate derivative of $\check{F}$: if $\check{F}^j$, or equivalently $\check{F}^j \circ {\varphi^j}^{-1}$ satisfies

$$\left[\frac{\partial \check{F}^j}{\partial L}(L^j(s))\right] \equiv \left[\frac{\partial \check{F}^j \circ (\varphi^j)^{-1}}{\partial y^j}(\varphi^j(L^j(s)))\right] = \Phi^j(s), \tag{9}$$

then (6) and (7) are satisfied. Note that this determination of $\Phi^j(s)$ establishes the regularity of $(L^j(s), t^j(s))$ in $G_{++} \times [0,1]$ because (9) implies that the matrix

$$\left[\frac{\partial \check{z}^1}{\partial L}(L^j(s)) + t^j(s)\frac{\partial \check{F}^j}{\partial L}(L^j(s)), \check{F}(L^j(s))\right]$$

maps $\left(\begin{smallmatrix}\alpha_l^j(s)\\0\end{smallmatrix}\right)$ to $\beta_l^j(s)$, $l = 2, \dots, k$ and $\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$ to $\check{F}(L^j(s))$ at any $s$.

We want to obtain an $\check{F}^j : B^j \to R^k$ whose value and derivative at $L^j(s)$ are respectively $\Psi^j(s)$ and $\Phi^j(s)$. This is achieved by the next lemma.

**Lemma 7.** Let $h : [0,1] \to R^n$ be a smooth injective map and let $\psi : [0,1] \to R$ and $\phi : [0,1] \to R^n$ be a smooth map. If

$$[\phi_1(s), \dots, \phi_n(s)]\begin{bmatrix}\frac{\partial h_1}{\partial s}(s)\\ \vdots\\ \frac{\partial h_n}{\partial s}(s)\end{bmatrix} = \frac{\partial \psi}{\partial s}(s), \tag{10}$$

then there exists a function $f : R^n \to R$ defined in a neighborhood of $\bigcup_s h(s)$ such that

$$f(h(s)) = \psi(s), \tag{11}$$

$$\left[\frac{\partial f}{\partial x_1}(h(s)), \dots, \frac{\partial f}{\partial x_n}(h(s))\right] = [\phi_1(s), \dots, \phi_n(s)], \tag{12}$$

where $x = (x_1, \dots, x_n)$ is the coordinate function of $R^n$.

**Proof.** See Appendix. ■

What this lemma maintains should be clear: the existence of a function $f$ whose value and derivative at $h(s)$ are respectively $\psi(s)$ and $\phi(s)$. Note that (10), which is obtained by differentiating (11) with respect to $s$ and substituting (12) into it, is a necessary condition for the existence of the function $f$ satisfying (11) and (12).

We can construct each $F_l^j \circ (\varphi^j)^{-1}$, $l = 1, \ldots, k$ by applying the above lemma. Of course, $n$, $h(s)$, $\psi(s)$, $\phi(s)$, and $x$ in Lemma 7 correspond to $k(M-k)$, $\varphi^j(L^j(s))$, $l$-th element of $\Psi^j(s)$, $l$-th row of $\Phi^j(s)$, and $y^j$, respectively. It is also easy to observe that (10) in the lemma corresponds to (7) in our problem. Thus we obtain $\check{F}^j$, hence $F^j$ satisfying (F1) and (F2).

Since our paths satisfy (viii) and (xi), we can make each $B^j$ sufficiently small so that $\overline{B^j} \bigcap \overline{B^{i'}} = \emptyset$. Then it is immediate to extend $F^j$ to $F$ defined on $G_{++}^k(R^M)$ satisfying Walras' law and $F|B^j = F^j$. This ends the construction of $F$ satisfying (F1) and (F2).

We have not yet considered the condition (F3). All we have to do is to control the index at one end point, say at $(p^j(0), t^j(0))$, of paths $j = 2, \ldots, r$. This is easily achieved by suitably taking $\beta_l^j(s)$ or $\alpha_l^j(s)$, $l = 2, \ldots, k$, we used to determine $\Phi^j(s)$. For example, take $-\beta_k^j(0)$ in place of $\beta_k^j(0)$ leaving other $\beta_l^j(0)$, $l = 2, \ldots, k-1$ and $\alpha_l^j(0)$, $l = 2, \ldots, k$ unchanged, and construct $F^j$. Then the index at $(p^j(0), t^j(0))$ changes its signs. This statement might need some explanation.

First, remember that the index of $H_F$ at $(p,t) \in H_F^{-1}(0)$ is determined according to whether the map $dH_{F,(p,t)}(\cdot, 0) : T_p\Delta \to p^\perp$ preserves the orientation. Second, also bear in mind that our determination of $\Phi^j(s)$ depends on our choice of $\alpha_2^j(s), \ldots, \alpha_k^j(s)$ on $dL_{p^j(s)}(T\Delta)$ and $\beta_2^j(s), \ldots, \beta_k^j(s)$ in $R^k$ and the map $d\tilde{H}_{F,(p,t)}(\cdot, 0)$ satisfies

$$d\tilde{H}_F(\frac{\partial p^j}{\partial s}(s), 0) = -F^j(L^j(s))\frac{\partial t^j}{\partial s}(s), \tag{13}$$

and

$$d\tilde{H}_F(\gamma_l^j(s), 0) = \tilde{\beta}_l^j(s),\ l = 2, \ldots, k, \tag{14}$$

where $\gamma_l^j(s) \in T_{p^j(s)}\Delta$ is the vector such that $dL_{p^j(s)}(\gamma_l^j(s)) = \alpha_l^j(s)$, $s \neq \bar{s}^j$, and $\tilde{\beta}_l^j(s) \in L(p)$ is the vector determined from $\beta_l^j(s)$ by $\check{\tilde{\beta}}_l^j(s) = \beta_l^j(s)$, $s \neq \bar{s}^j$. Since $\tilde{H}_F(p^j(s), t^j(s)) = H(p^j(s), t^j(s)) = 0$, we have

$$dH_F(\frac{\partial p^j}{\partial s}(s), 0) = -F^j(L^j(s))\frac{\partial t^j}{\partial s}(s), \tag{15}$$

and, as shown in Claim 5 in Appendix, there exists $\gamma_l^{\prime j}(s)$ such that

$$dH_F(\gamma_l^{\prime j}(s), 0) = \tilde{\beta}_l^j(s),\ l = 2, \ldots, k. \tag{16}$$

Note that, at $s \neq \bar{s}^j$, (15) and (16) are all on $L(p^j(s))$. As in the proof of Claim 6 in Appendix, there are further independent vectors $v_l^j(s)$, $l = 1, \dots, M-k-1$ on $T_{p^j(s)}\Delta$ and

$$dH_F(v_l^j(s), 0) = w_l^j(s), l = 1, \dots, M-k-1, \tag{17}$$

where $w_l^j(s)$, $j = 1, \dots, M-k-1$ are independent and not in $L(p^j(s))$.

Summarizing (15)–(17), at $(p^j(0), t^j(0))$ for example, the $dH_F(\cdot, 0)$ maps independent $M-1$ vectors, $\frac{\partial p^j}{\partial s}(0)$, $\gamma_l^{\prime j}(0)$, $l = 2, \dots, k$, and $v_l^j(0)$, $l = 1, \dots, M-k-1$, respectively to independent $M-1$ vectors, $-F^j(L^j(0))\frac{\partial t^j}{\partial s}(0)$, $\tilde{\beta}_l^j(0)$, $l = 2, \dots, k$, and $w_l^j(0)$, $l = 1, \dots, M-k-1$.

Therefore, if we take $-\beta_k^j(0)$ in the place of $\beta_k^j(0)$, with the others remaining unchanged, and determine new $\Phi^j$ and construct new $F$, then, with this new $F$, $dH_F(\cdot, 0)$ maps $\gamma_k^{\prime j}(0)$ to $-\tilde{\beta}_k^j(0)$ not to $\tilde{\beta}_k^j(0)$, hence the orientation is reversed and the index of $H_F$ at $(p^j(0), t^j(0))$ has the sign different from the earlier one.

## 6.4 Modification of excess demand of constrained agents

We first modify $F$ to a continuous function $\bar{F}$, which equals $F$ in a neighborhood of $\bigcup_{j,s} L^j(s)$ and satisfies not only (W') but also (BB') and (BC'). Let $B \subset G_{++}$ be an open subset that includes $\bigcup_{j,s} L^j(s)$. Take a sufficiently small $\epsilon$ such that $\overline{B} \subset G_\epsilon$ and define $\bar{F} : G_{++} \to R^M$ by

$$\bar{F}(L) = \eta(L)F(L) + (1 - \eta(L))proj_L(\mathcal{A}^0),$$

where $\eta$ is a smooth function such that $\eta(L) = 1$ for $L \in B$ and $\eta(L) = 0$ for $L \in G_{++} \backslash G_\epsilon$.

We next modify $\bar{F}$ to $Z$ so that $\tilde{z}^1(L) + tZ(L) = 0$ does not have solutions other that $(L^j(s), t^j(s))$ in $G_{++} \times [0,1]$. Of course, we construct $Z$ so that it is continuous, equals to $\bar{F}$, hence to $F$, in a neighborhood of the paths $\bigcup_{j,s} L^j(s)$, and satisfies (W')(BB')(BC'). It is not difficult to observe that if $\tilde{z}^1(L) + tZ(L) = 0$ does not have such solutions, that is, $(L^j(s), t^j(s))$ are all the solutions of $\tilde{z}^1(L) + tZ(L) = 0$, then $(p^j(s), t^j(s))$ are all the solutions of $z^1(p) + tZ(L(p)) = 0$ in $\Delta^g \times [0,1]$.

We let $D \subset G_{++} \times [0,1]$ be the set of solutions $(L, t)$ satisfying $\tilde{z}^1(L) + t\bar{F}(L) = 0$ other than $(L^j(s), t^j(s))$, $s \in [0,1]$, $j = 1, \dots, r$ and let $D'$ be the projection of $D$ onto $G_{++}$.

We first show that $D'$ and $\bigcup_{j,s} L^j(s)$ are disjoint. Since we have established the regularity of the path $(L^j(s), t^j(s))$ in $G_{++} \times [0,1]$, if there exists $L' \in D' \bigcap (\bigcup_{j,s} L^j(s))$, then there exists $(L', t')$ satisfying $\tilde{z}^1(L') + t'\bar{F}(L') = 0$ and $s'$ and $j$ such that $L' =$

$L^j(s')$ and $t \neq t^j(s')$. Since $\bar{F}$ equals to $F$ in a neighborhood of $L^j(s')$, this means $\tilde{z}^1(L^j(s')) + t'F(L^j(s')) = 0$ on the one hand, and $\tilde{z}^1(L^j(s')) + t^j(s')F(L^j(s')) = 0$, $t^j(s') \neq t'$ on the other hand. Then $\tilde{z}^1(L^j(s')) = 0$ and $F(L^j(s')) = 0$ hold. These implies that $\tilde{z}^j(L^j(s')) + tF(L^j(s')) = 0$ holds for any $t$. Then, for this path $j$, $L^j(s) = L^j(s')$ for any $s \in [0, 1]$. This contradicts our assumption (ix) that supports (I8).

Since $D$ and $D'$ are closed sets because of the continuity of $\tilde{z}^1$ and $\bar{F}$, let $C \subset G_\epsilon$ be an open subset including $D'$ in its interior and disjoint from $\bigcup_{j,s} L^j(s)$. We define a smooth function $\phi : G_{++} \to (0, 1]$ so that $\phi(L) = 1$ for $L \notin C$ and

$$D \subset \{(L, t) \in C \times [0, 1] | t > \phi(L)\}.$$

We define $Z(L) = \phi(L)\bar{F}(L)$. Then, for $L \notin C$, $\tilde{z}^1(L) + tZ(L) = \tilde{z}^1(L) + t\bar{F}(L)$, and $(L^j(s), t^j(s))$, $s \in [0, 1]$, $j = 1, \ldots, r$ are all the solutions of $\tilde{z}^1(L) + tZ(L) = 0$ in $(G_{++} \setminus C) \times [0, 1]$. For $L \in C$, $\tilde{z}^1(L) + tZ(L) = \tilde{z}^1(L) + t\phi(L)\bar{F}(L)$. Since $t\phi(L) \leq \phi(L)$ for $t \in [0, 1]$, $\tilde{z}^1(L) + tZ(L) = 0$ does not have a solution in $C \times [0, 1]$ by the definition of $\phi$. Thus $(L^j(s), t^j(s))$, $s \in [0, 1]$, $j = 1, \ldots, r$ are all the solutions of $\tilde{z}^1(L) + tZ(L) = 0$ in $G_{++} \times [0, 1]$ as desired. In other words, $(p^j(s), t^j(s))$, $s \in [0, 1]$, $j = 1, \ldots, r$ are all the solutions of $z^1(p) + tZ(L(p)) = 0$ in $\Delta^g \times [0, 1]$. It is clear that $Z : G_{++} \to R^M$ is continuous, equals to $F$ in a neighborhood of the paths $\bigcup_{j,s} L^j(s)$ and satisfies (W')(BB')(BC').

## 6.5 Construction of constrained consumers

We pick up sufficiently small $\epsilon$ and, by applying Proposition 2 to the function $Z$, construct a set of a finite number of consumers $\{\precsim^i, \omega^i\}_{i \geq 2}$ that have the aggregate excess demand function $Z^*$ such that $Z^*|G_\epsilon = Z|G_\epsilon$ and $\sum_l Z^*_l(L) > 0$ for $L \in G_{++} \setminus G_\epsilon$. Without loss of generality, we assume we have made the $\epsilon$ sufficiently small so that $\sum_{l=1}^M \tilde{z}^1_l(L) > 0$.

If $L \in G_{++} \setminus G_\epsilon$, then $\sum_{l=1}^M \tilde{z}^1_l(L) > 0$ and $\sum_{l=1}^M Z^*_l(L) > 0$, hence $\tilde{z}(L) + tZ^*(L) = 0$ has no solution in $(G_{++} \setminus G_\epsilon) \times [0, 1]$.

From our construction of $F$, $\bar{F}$, $Z$ and $Z^*$, it is clear that $(L^j(s), t^j(s))$ are all the solutions of $\tilde{z}^1(L) + Z^*(L) = 0$ in $G_{++} \times [0, 1]$, hence $(p^j(s), t^j(s))$ are all the solutions of $z^1(p) + tZ^*(L(p)) = 0$ in $\Delta^g \times [0, 1]$. It is also clear that, since $Z^*$ equals to $F$ in a neighborhood of $L(p^j(s))$, $index \hat{H}_{Z^*}(p^j(s), t^j(s)) = index \hat{H}_F(p^j(s), t^j(s))$ at any $(p^j(s), t^j(s)) \in \Delta^g \times [0, 1]$.

Thus, the economy $\{\precsim^i \omega^i\}_{i \geq 1}$ realizes the path $(p^j(s), t^j(s))$, $s \in [0, 1]$, $j = 1, \ldots, r$, as its homotopy paths and the index of this economy is $\alpha$ as desired. This ends the proof of Theorem 5.

# A Appendix

## A.1 Proof of Lemma 6

As observed in Section 3.4, $\tilde{z}^1(L(p^j(s)) = z^1(p^j(s))$ for $s \neq \bar{s}^j$. Therefore, for such $s$, $\tilde{H}_F(p^j(s), t^j(s)) = \tilde{z}^1(L(p^j(s))) + t^i(s)F(L(p^j(s))) = z^1(p^j(s)) + t^j(s)F(L(p^j(s))) = H_F(p^j(s), t^j(s))$. Thus $\tilde{H}_F(p^j(s), t^j(s)) = 0$ implies $H_F(p^j(s), t^j(s)) = 0$.

We next consider the regularity of path. Note that, at $(p, t) \in \Delta^g \times [0, 1]$ such that $\tilde{H}_F(p, t|F) = 0$, $d\tilde{H}_{F,(p,t)}$ is a function into $L(p)$. We first consider $dL_p : T_p\Delta^g \to T_{L(p)}G^k_{++}(R^n)$, which is the differential of $L$ at $p \in \Delta^g$

**Claim 1.** [Bottazzi and Hens (1996)] $\pi \in L(p)^\perp \Leftrightarrow L(\pi) = L(p)$.

**Proof.** See Bottazzi and Hens (1996).

**Claim 2.** For $v \in T_p\Delta$, $dL_p(v) = 0 \Leftrightarrow v \in L(p)^\perp$.

**Proof.** Because of Claim 1, $v \in L(p)^\perp \Leftrightarrow p + tv \in L(p)^\perp$ for any $t \Leftrightarrow L(p + tv) = L(p)$ for any $t$. The last equation is $dL_p(v) = 0$ in other words. ■

**Claim 3.** If $v', v'' \in L(p)$ and $v' \neq v''$, then $dL_p(v') \neq dL_p(v'')$.

**Proof** Let $v', v'' \in L(p)$ and $v \neq v'$. If $v' - v'' \in L(p)$, then $v' - v'' \notin L(p)^\perp$, hence $L(p + t(v' - v'')) \neq L(p)$ for $t \neq 0$ as in Claim 1. That is, $dL_p(v' - v'') \neq 0$ and $dL_p(v') \neq dL_p(v'')$. ■

**Claim 4.** For non-zero $v \in T_p\Delta$ such that $v^T z^1(p) = 0$, $v^T dz^1_p(v) < 0$.

**Proof.** See Geanakoplos and Polemarchakis (1980)[Proposition A and Corollary A, p. 319] for example. The impact of a price change $v \in T_p\Delta$ at $p$ on the unconstrained excess demand function $z^1$ can be decomposed as

$$dz^1_p(v) = [\frac{\partial z^1}{\partial p}(p)]v = [K(p) + \nu(p)z^1(p)^T]v$$

where $[\frac{\partial z^1}{\partial p}(p)] = [K(p) + \nu(p)z^1(p)^T]$ denotes the Slutsky equation, $K(p)$ denotes the substitution matrix that is symmetry, negative semidefinite, and of rank $M - 1$, and $\nu(p)z^1(p)^T$ is the income effect matrix. Since $z^1(p)^T v = 0$ for our $v$, $v^T dz^1_p(v) = v^T K(p)v$. $v^T K(p)v$ is non-positive because $K(p)$ is negative semidefinite. However, since $K(p)p = 0$ and $K(p)$ is of rank $n - 1$, $v^T K(p)v \neq 0$ for non-zero $v$. ■

**Claim 5.** For any $(v, a) \in T(\Delta \times [0,1])$, there exists $v' \in T\Delta$ such that $d\tilde{H}_{F,(p,t)}(v, a) = dH_{F,(p,t)}(v', a)$ at $(p, t)$ such that $\tilde{H}_F(p, t) = H_F(p, t) = 0$.

It is sufficient for us to prove that, for any $(p', t) \in \Delta^g \times [0,1]$ in a neighborhood of $(p, t) \in H^{-1}(0)$, there exists $p'' \in \Delta^g$ in a neighborhood of $p$ such that $\tilde{H}(p', t) = H(p'', t)$.

Suppose $p'$ is given. Let $p''$ be the supporting price of the indifference surface of the first consumer at $\tilde{z}^1(L(p'))$. Then, it is clear that $\tilde{z}^1(L(p')) = z^1(p'')$. Since $p''$ is normal to $L(p')$, $L(p'') = L(p')$ as in Claim 1, hence $F(L(p') = F(L(p''))$. Thus, $\tilde{H}(p', t) = H(p, t)$.

When $\tilde{H}_F(p, t) = H_F(p, t) = 0$, $z^1(p) = \tilde{z}^1(L(p))$. That is, $p$ is the supporting price of the indifference surface of the first consumer at $\tilde{z}^1(L(p))$. Therefore if $p'$ is in a neighborhood of $p$, the $p''$ is also in a neighborhood of $p$. ■

**Claim 6.** At $(p, t) \in \Delta^g \times [0,1]$ such that $H_F(p, t) = \tilde{H}_F(p, t) = 0$, if $d\tilde{H}_F : T_{(p,t)}(\Delta \times [0,1]) \to L(p)$ is surjective, then $dH_F : T_{(p,t)}(\Delta \times [0,1]) \to p^\perp$ is surjective.

**Proof.** We suppose $d\tilde{H}_F$ is surjective at such $(p, t)$. From Claim 5, $L(p) = d\tilde{H}_F(T_{(p,t)}(\Delta \times [0,1])) \subset dH_F(T_{(p,t)}(\Delta \times [0,1]))$. We let $W_p = L(p)^\perp \bigcap p^\perp \subset T_p\Delta = p^\perp$, which is an $M - k - 1$-dimensional subspace. Since $L(p)$ is a $k$-dimensional subspace, all we have to show is that $dH_F(W_p \times \{0\})$ is an $M - k - 1$-dimensional subspace such that $dH(W_p \times \{0\}) \bigcap L(p) = \{0\}$.

Note that, for any $v \in W_p$, $v^T z^1(p) = 0$ because $v \in L(p)^\perp$ and $z^1(p) = -tF(L(p)) \in L(p)$. Also note that, for any $v \in W_p$, $dL(v) = 0$ as in Claim 2. Therefore, for any $v \in W_p$, $dH_F(v, 0) = dz^1(v) + tdF(dL(v)) = dz^1(v) \notin L(p)$ from the strict convexity of the preference ordering. Thus $dH_F(W \times \{0\}) \bigcap L(p) = \{0\}$.

On the other hand, for independent $v'$ and $v''$ in $T_p\Delta$, $dz^1(v')$ and $dz^1(v'')$ are independent. Thus $dH_F(W_p \times \{0\})$ is $M - k - 1$-dimensional. ■

## A.2 Proof of Lemma 7.

We let $\zeta : R^n \to R^n$ be a coordinate transformation defined in a neighborhood of $\bigcup_s h(s)$, which transforms $x = (x_1, \ldots, x_n)$ to $y = (y_1, \ldots, y_n) = (\zeta_1(x), \ldots, \zeta_n(x))$ so that

$$\zeta(h(s)) = (s, 0, \ldots, 0)$$

The existence of such $\zeta$ is clear from the injectivity of $h$. We define a function $\tilde{f}$ defined on a neighborhood of $\bigcup_s \zeta(h(s))$ so that $f(x) = \tilde{f} \circ \zeta(x)$ or, equivalently, $f \circ \zeta^{-1}(y) = \tilde{f}(y)$ and consider the conditions (11) and (12) on $\tilde{f}$. By substituting $x = h(s)$ into

$$\left[\frac{\partial f}{\partial x}(x)\right] = \left[\frac{\partial \tilde{f}}{\partial y}(\zeta(x))\right]\left[\frac{\partial \zeta}{\partial x}(x)\right],$$

(12) becomes

$$\left[\frac{\partial \tilde{f}}{\partial y}(s,0,\dots,0)\right] = [\phi(s)]\left[\frac{\partial \zeta}{\partial x}(x)\right]^{-1}.$$

We write the right-hand side of the above equation as $\tilde{\phi}(s) = [\tilde{\phi}_1(s),\dots,\tilde{\phi}_n(s)]$. Note that $\frac{\partial \tilde{f}}{\partial y_1}(s,0,\dots,0) = \tilde{\phi}_1(s) = \frac{\partial \psi}{\partial s}(s)$ because

$$\left[\frac{\partial \tilde{f}}{\partial y}(s,0,\dots,0)\right]\left[\frac{\partial \zeta}{\partial x}(h(x))\right]\left[\frac{\partial h}{\partial s}(s)\right] = \phi(s)\left[\frac{\partial h}{\partial s}(s)\right] = \frac{\partial \psi}{\partial s}(s)$$

and

$$\left[\frac{\partial \zeta}{\partial x}(h(x))\right]\left[\frac{\partial h}{\partial s}(s)\right] = \left[\frac{\partial \zeta}{\partial s}(s)\right] = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

On the other hand, (11) becomes $\tilde{f}(s,0,\dots,0) = \tilde{f}\circ\zeta(h(s)) = \psi(s)$

Now, it is easy to check that $\tilde{f}$ defined as

$$\tilde{f}(y) = \psi(y_1) + \tilde{\phi}_2(y_1)y_2 + \cdots + \tilde{\phi}_n(y_1)y_n$$

satisfies the above conditions. Therefore $f(x) = \tilde{f}(\zeta(x))$ satisfies the statement of the lemma. ■

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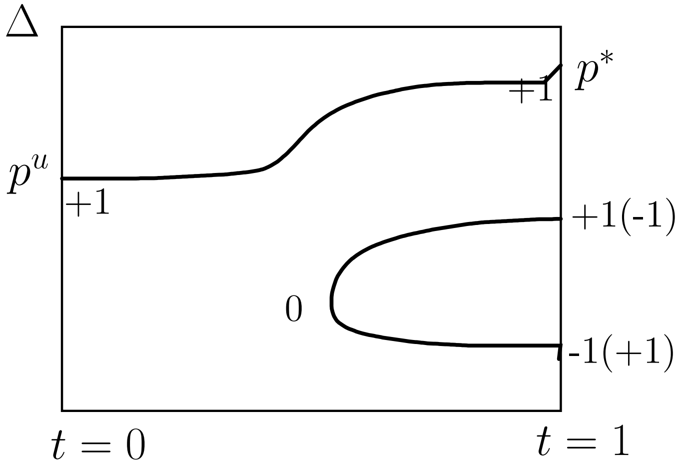

**Figure 1:** Example of homotopy paths in Arrow-Debreu model

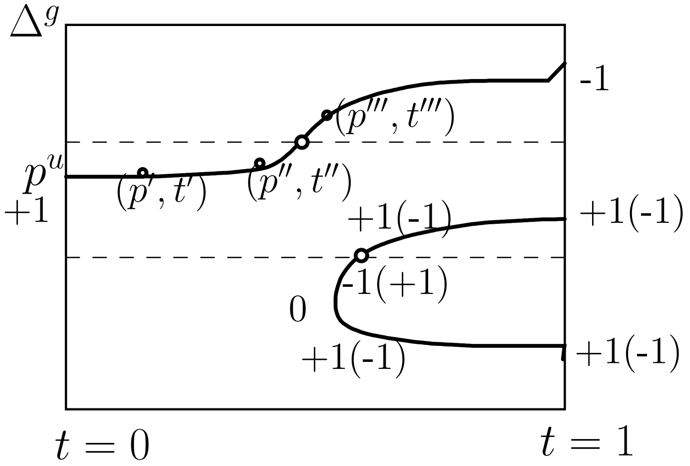

**Figure 2:** Example of homotopy paths in GEI model