Note on social choice allocation in exchange economies with Cobb-Douglas preferences *

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Abstract

In this note we show that in a pure exchange economy with two agents and a finite number of goods, there exists no strategy-proof, Pareto efficient and non-dictatorial social choice allocation function on any local Cobb-Douglas preference domain. This is a slight extension of a result proved by Hashimoto (2008).

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1 Introduction

Since the seminal work by Hurwicz (1972), the manipulability of an allocation mechanism in pure exchange economies has been intensively studied. After Zhou (1991) established that there exists no Pareto efficient, strategy-proof and non-dictatorial social choice function in an exchange economy with two agents having classical (i.e. continuous, strictly monotonic and strictly convex) preferences, many authors analyzed whether the impossibility result holds on further restricted preference domains (Schummer (1997), Ju (2003), Nicolò (2004)). Recently Hashimoto (2008) proved that there exists no Pareto efficient, strategy-proof and non-dictatorial social choice function in a two-agent exchange economy with Cobb-Douglas preferences.

In this note we sophisticate Hashimoto's approach and prove that his technical assumption that the preference domains of the both agents include an identical preference

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is redundant. We show that there exists no Pareto efficient, strategy-proof, and non-dictatorial social choice function in a two-agent exchange economy where the two agents might have different Cobb-Douglas preference domains.

An interesting finding by Hashimoto's approach is that under a Pareto efficient and strategy-proof social choice function, a change of one agent's preference should not affect the utility level of the other agent, and hence the allocation given by the social choice function can be specified. Our simple proof would highlight that this interesting property would hold not only on Cobb-Douglas domains but also on more general domains. This is a useful technique to investigate the allocation given by a Pareto efficient and strategy-proof social mechanism. See Momi (2011) for an application of this approach, where the social choice allocation in an many-agent economy with smooth and homothetic preferences is investigated.

2 The model

We consider pure exchange economies with two agents indexed by $i = 1, 2$ and L goods indexed by $l = 1, \dots, L$ ($L \geq 2$). The consumption set of the each agent is R^L_+. A consumption bundle of agent i is $x_i = (x_{i1}, \dots, x_{iL}) \in R^L_+$ where x_{il} is his consumption of the l-th good. An allocation is $x = (x_1, x_2) \in R^{2L}_+$. The total endowment of goods is $\omega = (\omega_1, \dots, \omega_L) \in R^L_{++}$. The set of feasible allocation is $X = \{(x_1, x_2) \in R^{2L}_+ | x_1 + x_2 = \omega\}$.

Each agent i has a preference represented by a Cobb-Douglas utility function U_i on the consumption space R^L_+:

$$U_i(x; a^i) = x_1^{a_{i1}} \cdots x_L^{a_{iL}}$$

where $a_i = (a_{i1}, \dots, a_{iL}) \in R^L_{++}$ is the parameter defining the utility function. Clearly a^i can be identified with the utility function; and hence with the preference represented by the utility function. If a^i equals to $\bar{a}^i$ up to normalization ($a^i = t\bar{a}^i$ with $t \in R_{++}$), the preference defined by a^i equals to one by $\bar{a}^i$. A preference profile is a list of preferences of agents $a = (a_1, a_2) \in R^{2L}_{++}$. We also write $a = (a_i, a_j)$ to denote the preference profile where agent i's preference is a_i and j's a_j ($i, j = 1, 2$ and $i \neq j$). To deal with the case where we are interested in restricted set of the preferences, we let $A_i \subset R^L_{++}$ denote the set of a_i we are concerned and write $A = A_1 \times A_2$

A social choice function $f : A \to X$ is a map from a preference profile to an allocation. Let $f_i(a) = (f_{i1}(a), \dots, f_{iL}(a))$ denote the consumption bundle allocated to agent i by f at a.

Definition 1. An allocation $x \in X$ is Pareto efficient for a if there exists no $\bar{x} \in X$ such that $U_i(\bar{x}_i; a_i) \geq U_i(x_i; a_i)$ for all $i = 1, 2$, and $U(\bar{x}_j; a_j) > U(x_j; a_j)$ for some $j = 1, 2$. A social choice function $f : A \to X$ is Pareto efficient if $f(a)$ is a Pareto efficient allocation for any $a \in A$.

Definition 2. A social choice function $f : A \to X$ is strategy-proof if $U_i(f_i(a_i, a_j); a_i) \geq U_i(f_i(a_i', a_j); a_i)$ for any $i, j = 1, 2$, $i \neq j$, any $(a_i, a_j) \in A$, and any $a_i' \in A_i$.

Definition 3. A social choice function f is dictatorial if there exists some agent i such that $f^i(a) = \omega$ for any $a \in A$.

3 Result and proof

Proposition 1. For any open sets A_1 and A_2, there exists no social choice function $f : A^1 \times A^2 \to X$ that is Pareto efficient, strategy-proof, and non-dictatorial.

The proof is essentially the same as the proof by Hashimoto (2008). We first show that $f(\cdot, a_j) : A_i \to X$ is a continuous function of a_i. Note that this does not generally imply that $f : A_1 \times A_2 \to X$ is a continuous function

Lemma 1. If $f : A \to X$ is a Pareto efficient and strategy-proof social choice function, then $f(\cdot, a_j) : A_i \to X$ is a continuous function for any $a_j \in A_j$, $i, j = 1, 2$, $(i \neq j)$.

Proof. We arbitrarily fix $a_2 \in A_2$ and show that the function $f(\cdot, a_2) : A_1 \to X$ is continuous.

We suppose $a_1 \to \bar{a}_1 \in A_1$. Since X is compact, $f(a_1, a_2)$ converges as $a_1 \to \bar{a}_1$. We let $f(a_1, a_2) \to \bar{x}$ as $a_1 \to \bar{a}_1$. All we have to show is that $\bar{x} = f(\bar{a}_1, a_2)$.

Since f is strategy-proof, $U_1(f_1(a_1, a_2); a_1) \geq U_1(f_1(\bar{a}_1, a_2); a_1)$ holds for any a_1. Especially at the limit of $a_1 \to \bar{a}$, $U_1(\bar{x}_1; \bar{a}_1) \geq U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)$ holds. If this equation holds with strict inequality, then the consumer could be better off by reporting $\tilde{a}_1$ which is sufficiently close to $\bar{a}_1$ when his true preference is $\bar{a}_1$, because $f_1(\tilde{a}_1, a_2)$ is close to $\bar{x}_1$, and hence $U_1(f_1(\tilde{a}_1, a_2); \bar{a}_1)$ is close to $U_1(\bar{x}_1; \bar{a}_1)$. This violates the strategy-proofness of f. Therefore the equation should hold with equality: $U_1(\bar{x}_1; \bar{a}_1) = U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)$.

We next show that $\bar{x}$ should be a Pareto efficient allocation with respect to preferences $\bar{a}_1$ and a_2. Suppose that $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is not Pareto efficient. Then in the economy with strictly convex preferences $\bar{a}_1$ and $\bar{a}_2$ there exists $x' = (x_1', x_2') \in X$ such that $U_1(x_1'; \bar{a}_1) > U_1(\bar{x}_1; \bar{a}_1)$ and $U_2(x_2'; a_2) > U_2(\bar{x}_2; a_2)$. When $\tilde{a}_1$ is sufficiently close to $\bar{a}_1$, $f(\tilde{a}_1, a_2)$ is sufficiently close to $\bar{x}$ and $U_1(f_1(\tilde{a}_1, a_2); \tilde{a}_1)$ is sufficiently close to $U_1(f_1(\tilde{a}_1, a_2); \bar{a}_1)$. Therefore $U_1(x_1'; \tilde{a}_1) > U_1(f_1(\tilde{a}_1, a_2); \tilde{a}_1)$ and $U_2(x_2'; a_2) > U_2(f_2(\tilde{a}_1, a_2); a_2)$ hold. This violates the Pareto efficiency of f.

It is easy to observe that in the Edgeworth Box the set of Pareto efficient allocations intersects each consumer's one indifference surface only once. Therefore our observations that $U(\bar{x}_1;\bar{a}_1) = U(f(\bar{a}_1, a_2);\bar{a}_1)$ and $\bar{x}$ and $f(\bar{a}_1, a_2)$ are both Pareto efficient allocations imply that $\bar{x} = f(\bar{a}_1, a_2)$. ■

We next show that any changes of a agent's preference should not affect the utility level of the other agent.

Lemma 2. If $f : A \to X$ is a Pareto efficient and strategy-proof social choice function, then $U_j(f_j(a_i, a_j); a_j) = U_j(f_j(a'_i, a_j); a_j)$ for any $i, j = 1, 2$, $(i \neq j)$ and any $a_i, a'_i \in A_i$ and $a_j \in A_j$.

Proof. We set $i = 1$ and $j = 2$ and prove the lemma for $a_1 = (a_{11}, a_{12}, \ldots, a_{1L})$ and $a'_1 = (a'_{11}, a_{12}, \ldots, a_{1L})$. That is, we prove that the utility level of the consumer 2 is not affected when a_{11} is changed. The discussion is symmetric for other elements $a_{12}, \ldots, a_{1L}$. Since these changes of each element sums up to any changes of the parameter a_1, this is sufficient as the proof of the lemma.

Since $(a_{12}, \ldots, a_{1L})$ and a_2 are fixed in the following discussions, we simply write $f_1(a_{11})$ and $f_2(a_{11})$ to denote the consumptions given by f at (a_1, a_2).

We suppose that there exists a'_{11} and a''_{11} such that $U_2(f_2(a'_{11}); a_2) \neq U_2(f_2(a''_{11}); a_2)$. Without loss of generality we assume $a'_{11} < a''_{11}$.

We first consider the case where $U_2(f_2(a'_{11}); a_2) > U_2(f_2(a''_{11}); a_2)$. Note that $U_2(f_2(a_{11}); a_2)$ is a continuous function of a_{11} by Lemma 1 proved above. Then there exists $\bar{a}_{11} \in (a'_{11}, a''_{11})$ and a sequence $\{\epsilon_n\}$ which converges to 0 from the right hand side (i.e. $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$) such that

$$\lim_{n\to\infty} \frac{U_2(f_2(\bar{a}_{11} + \epsilon_n); a_2) - U_2(f_2(\bar{a}_{11}); a_2)}{\epsilon_n} < 0.$$[1]

Since the utility function $U_2(\cdot; a_2)$ is differentiable, the equation implies

$$\sum_{l=1}^{L} \frac{\partial U_2(f_2(\bar{a}_{11}); a_2)}{\partial x_{2l}} \lim_{n\to\infty} \frac{f_{2l}(\bar{a}_{11} + \epsilon_n) - f_{2l}(\bar{a}_{11})}{\epsilon_n} < 0.$$

Since f is Pareto efficient, $f_2(a_{11}) = \omega - f(a_{11})$ holds for any a_{11} and $(\frac{\partial U_2(f_2(a_{11});a_2)}{\partial x_{21}}, \ldots, \frac{\partial U_2(f_2(a_{11});a_2)}{\partial x_{2L}})$ is parallel to $(\frac{\partial U_1(f_1(a_{11});a_1)}{\partial x_{11}}, \ldots, \frac{\partial U_1(f_1(a_{11});a_1)}{\partial x_{1L}})$. Therefore we have

$$\sum_{l=1}^{L} \frac{\partial U_1(f_1(\bar{a}_{11}); \bar{a}_1)}{\partial x_{1l}} \lim_{n\to\infty} \frac{f_{1l}(\bar{a}_{11} + \epsilon_n) - f_{1l}(\bar{a}_{11})}{\epsilon_n} > 0,$$

hence,

$$\lim_{n\to\infty} \frac{U_1(f_1(\bar{a}_{11} + \epsilon_n); \bar{a}_1) - U_1(f_1(\bar{a}_{11}); \bar{a}_1)}{\epsilon_n} > 0,$$

[1] To the contrary, suppose that $\lim_{n\to\infty} \frac{U_2(f_2(\bar{a}_{11}+\epsilon_n);a_2) - U_2(f_2(\bar{a}_{11});a_2)}{\epsilon_n} > 0$ for any $\bar{a}_{11} \in (a'_{11}, a''_{11})$ and any sequence $\{\epsilon_n\}$ converging 0 from right hand side. It clearly contradicts to that $U_2(f(\cdot); a_2)$ is a continuous function and $U_2(f_2(a'_{11}); a_2) > U_2(f_2(a''_{11}); a_2)$.

where $\bar{a}_1 = (\bar{a}_{11}, a_{12} \ldots, a_{1L})$. This implies $U_1(f_1(\bar{a}_{11} + \epsilon_n); \bar{a}_1) > U_1(f_1(\bar{a}_{11}); \bar{a}_1)$ with sufficiently large n because $\epsilon_n > 0$. This violates the strategy-proofness of f because consumer 1 is better off by announcing $(\bar{a}_{11} + \epsilon_n, a_{12}, \ldots, a_{1L})$ when his true preference is $\bar{a}_1$.

Next, we consider the case where $U_2(f_2(a'_{11}); a_2) < U_2(f_2(a''_{11}); a_2)$. Then there exists $\bar{a}_{11} \in (a'_{11}, a''_{11})$ and a sequence $\{\epsilon_n\}$ which converges to 0 from the left hand side, $\epsilon_n < 0$ and $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{U_2(f_2(\bar{a}_{11} + \epsilon_n); a_2) - U_2(f_2(\bar{a}_{11}); a_2)}{\epsilon_n} > 0.$$

By the same discussion, we have

$$\lim_{n\to\infty} \frac{U_1(f_1(\bar{a}_{11} + \epsilon_n); \bar{a}_1) - U_1(f_1(\bar{a}_{11}); \bar{a}_1)}{\epsilon_n} < 0.$$

This implies $U_1(f_1(\bar{a}_{11} + \epsilon_n); \bar{a}_1) > U_1(f_1(\bar{a}_{11}); \bar{a}_1)$ with sufficiently large n because $\epsilon_n < 0$. This again violates the strategy-proofness of f. ∎

Remark that we need continuity and strict convexity of preferences in Lemma 1 and differentiability in Lemma 2.

Proof of Proposition 1. To the contrary, suppose that f is a Pareto efficient, strategy-proof and non-dictatorial social choice function. We can select $a_1 \in A_1$ and $a_2 \in A_2$ such that $f(a_1, a_2)$ is in the interior of X.

Choose any $\bar{a}_1$ which is sufficiently close to a_1. See Figure where the Edgeworth Box of the economy is drawn. As proved in Lemma 2, $f_2(\bar{a}_1, a_2)$ is indifferent to $f_2(a_1, a_2)$ with respect to the preference a_2. In the Edgeworth box, consider a ray starting from the vertex of consumer 2 ($(x_1, x_2) = (\omega, 0)$) and passing through $f(a_1, a_2)$. Also consider the indifference set of consumer 1 with preference $\bar{a}_1$ at $f_1(\bar{a}_1, a_2)$: $\{x_1 \in R^L_+ | U_1(x_1; \bar{a}_1) = U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)\}$. Let $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ denote the intersection of this ray and this indifference set in the Edgeworth box. It is easy to observe that the intersection is determined uniquely. It is also easy to observe that $\bar{x}_2 = t f_2(a_1, a_2)$ with some $0 < t < 1$.

We select a parameter $\bar{a}_2$ so that the gradient vector $(\frac{\partial U_2(\bar{x}_2;\bar{a}_2)}{\partial x_{21}}, \ldots, \frac{\partial U_2(\bar{x}_2;\bar{a}_2)}{\partial x_{2L}})$ of the preference $\bar{a}_2$ at $\bar{x}_2$ is parallel to the gradient vector $(\frac{\partial U_1(\bar{x}_1;\bar{a}_1)}{\partial x_{11}}, \ldots, \frac{\partial U_1(\bar{x}_1;\bar{a}_1)}{\partial x_{1L}})$ of $\bar{a}_1$ at x_1, that is, we select a preference a_2 so that $\bar{x}$ is a Pareto efficient allocation with respect to $\bar{a}_1$ and $\bar{a}_2$. Note that when $\bar{a}_1$ is sufficiently close to a_1, $\bar{x}$ is close to $f(a_1, a_2)$, and hence the gradient vector of the preference $\bar{a}_1$ at $\bar{x}_1$ is close to that of a_1 at $f_1(a_1, a_2)$. Therefore we can find such an $\bar{a}_2$ in the neighborhood of a_2. Because of Lemma 2, $f(\bar{a}_1, \bar{a}_2)$ should be on the consumer 1's indifference set $\{x_1 \in R^L_+ | U_1(x_1; \bar{a}_1) = U_1(f_1(\bar{a}_1, a_2); \bar{a}_1)\}$. Since the indifference set intersects the set of Pareto optimal allocations with preferences $\bar{a}_1$ and $\bar{a}_2$ only once, we have $f(\bar{a}_1, \bar{a}_2) = \bar{x}$.

Finally consider the allocation $f(a_1, \bar{a}_2)$. Observe that $f_2(a_1, \bar{a}_2)$ is indifferent to $\bar{x}_2$ with respect to the preference $\bar{a}_2$ because of Lemma 2 and that $f_2(a_1, a_2)$ is preferred to $\bar{x}_2$ with respect to any preferences. Thus $U_2(f_2(a_1, a_2); \bar{a}_2) > U_2(f_2(a_1, \bar{a}_2); \bar{a}_2)$, which violates the strategy-proofness of f. ■

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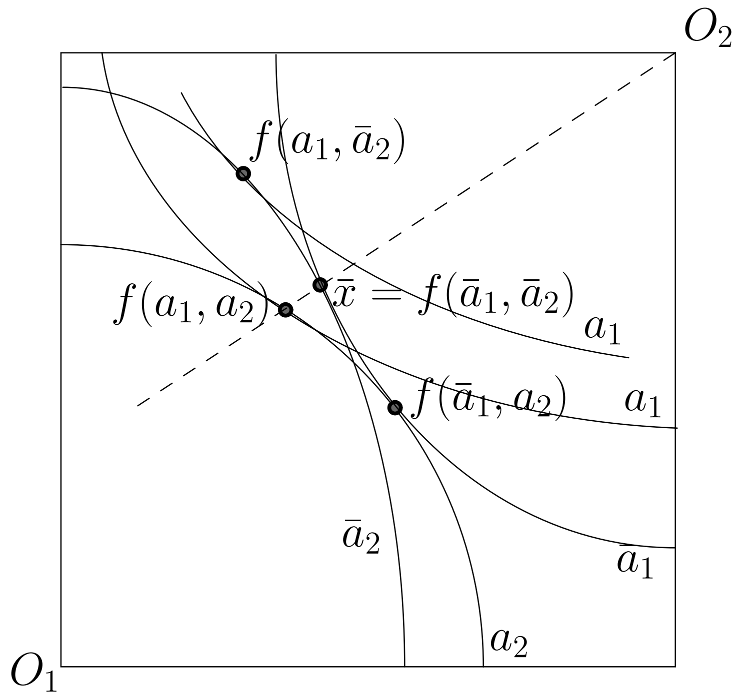

Figure The Edgeworth Box