Geometry of a Quartic Thue Equation

by Ryotaro Okazaki at Doshisha University

December 01, 2001

1 Introduction

Theorem 1 *Assume*

$$f(X,Y) \in \mathbf{Z}[X,Y]$$

is a homogeneous irreducible quartic form. Assume also $f(X,Y)$ splits completely in a totally real number field. Let $\mathcal{R}(f)$ be the number of integer points on the Thue curve

$$\mathcal{T} : f(x,y) = \pm 1,$$

where $\pm(x,y)$ are counted as one integer point. Then, $\mathcal{R}(f) \leq 14$ provided the discriminant D of f is larger than a computable constant D_0.

(The computable constant D_0 is under calculation and is expected to be less than 10^{2001}.)

To achieve this theorem, we parametrize the Thue curve $\mathcal{T}$ by $t \in \mathbf{P}^1(\mathbf{R})$ and investigate real projective geometry of t and the configuration of the four roots of $f(X,1)$. We will find a canonical transcendental curve $\mathcal{C} = \phi(\mathbf{P}^1(\mathbf{R}))$ that interfaces the Thue curve $\mathcal{T}$ and the unit group $\mathcal{E}$. The connection of the continuous curve $\mathcal{T}$ and the discrete lattice $\mathfrak{E} = \log \mathcal{E}$ implies an exponential gap principle of geometric nature. The exponential gap principle has effect on the so-called large solutions of the Thue equation $f(x,y) = \pm 1$.

For better understanding of the so-called small solutions, we introduce a notion of admissible transformations that preserve discreteness. By studying solid geometry of the curve $\mathcal{C}$, we will find a way of bisecting the interval between roots. We use admissible transformation to normalize the roots by mapping a pair of bisectors to 0 and ∞. After normalization, the polynomial $f(X,Y)$ can be go outside of $\mathbf{Z}[X,Y]$. However, we will find the symmetry of $\mathcal{C}$ through this normalization. We will then wrap $\mathcal{C}$ by 5 cylinders, i.e., 4 asymptotic cylinders and a cylinder of the bridge. The exponential gap principle will be made explicit in the asymptotic cylinders. Indeed, we will use an inscribed parallelotope of the cylinder for the bridge. We can estimate the integer points which correspond to points in this parallelotope.

Combining the exponential gap principle with some lower bound of logarithmic form, we will establish the Theorem.

2 Real Projective Geometry

Let

$$\mathcal{A} = \{\alpha_1 < \alpha_2 < \ldots < \alpha_n\}, \quad n \geq 4,$$

be a given configuration of points. Let

$$f(X, Y) = f(\mathcal{A}, X, Y) = \prod_{i=1}^{n} (X - Y\alpha_i).$$

And consider the curve

$$\mathcal{T} : |f(x, y)| = 1.$$

The projective point

$$t = \frac{x}{y} \in \mathbf{P}^1(\mathbf{R})$$

parameterizes $\mathcal{T}/\{\pm 1\}$ by

$$\begin{cases} y(t) &= & y(\mathcal{A}, t) &= & |f(t)|^{-1/n}, \\ x(t) &= & x(\mathcal{A}, t) &= & t\, y(t), \end{cases}$$

where $f(t) = f(\mathcal{A}; t) = f(t, 1)$.

A projective transformation of $t \in \mathbf{P}^1(\mathbf{R})$ is defined by:

$$G = (g_{ij}) \in GL_2(\mathbf{R}) : t \longmapsto G\langle t \rangle = \frac{g_{11}t + g_{12}}{g_{21}t + g_{22}}.$$

We adopt the following convention:

$$\tilde{t} = G\langle t \rangle, \quad \tilde{\alpha}_i = G\langle \alpha_i \rangle, \quad \tilde{\mathcal{A}} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}; \quad \tilde{x} = x(\tilde{\mathcal{A}}, \tilde{t}), \quad \tilde{y} = y(\tilde{\mathcal{A}}, \tilde{t}).$$

The difference of two points $u, u' \in \mathbf{R} \subset \mathbf{P}^1(\mathbf{R})$ is transformed by the following rule:

$$\tilde{u} - \tilde{u}' = \frac{\det \begin{pmatrix} g_{11}u + g_{12} & g_{11}u' + g_{12} \\ g_{21}u + g_{22} & g_{21}u' + g_{22} \end{pmatrix}}{\chi(G,u)\chi(G,u')} = \frac{(u - u')\det G}{\chi(G,u)\chi(G,u')},$$

where

$$\chi(G,t) = g_{21}t + g_{22}.$$

Change of variable is more common in the study of Thue equation than projective transformation is. For comparing the two aspects, we write

$$x - y\alpha_i = \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}; \quad f(x,y) = \prod_{i=1}^{n} \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}.$$

We define

$$\tilde{f}(X,Y) = f(\tilde{A}, X, Y) = \prod_{i=1}^{n} \det \begin{pmatrix} X & \tilde{\alpha}_i \\ Y & 1 \end{pmatrix}$$

When

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = G \begin{pmatrix} x \\ y \end{pmatrix},$$

we have

$$\prod_{i=1}^{n} \det \begin{pmatrix} x_1 & \tilde{\alpha}_i \\ y_1 & 1 \end{pmatrix} = \frac{\det G^n}{\prod_{i=1}^{n} \chi(G,\alpha_i)} \prod_{i=1}^{n} \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}.$$

Therefore, we get the following condition

$$G \begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \Longleftrightarrow \left| \frac{\det G^n}{\prod_{i=1}^{n} \chi(G,\alpha_i)} \right| = \left| \frac{\tilde{f}(\tilde{x},\tilde{y})}{f(x,y)} \right| = 1 \tag{1}$$

of compatibility.

3 Transcendental Curve $\mathcal{C}$

Define

$$\phi_m(t) = \phi_m(\mathcal{A}, t) = \log \left| \frac{D^{1/n(n-2)}(x - y\alpha_m)}{|f'(\alpha_m)|^{1/(n-2)}} \right| \tag{2}$$

with

$$D = D(\mathcal{A}) = \prod_{i \neq j} |\alpha_i - \alpha_j|\,.$$

Then, each $\phi_m(t)$ is invariant under the action of $GL_2(\mathbf{R})$ (on t and α_i's). We now define

$$\phi(t) = \phi(\mathcal{A}, t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t)).$$

Then, $\phi(t)$ is invariant upto permutation of coordinates under the action of $GL_2(\mathbf{R})$. An immediate consequence is invariance of

$$\phi(t') - \phi(t).$$

Deeper consequences come from geometry of the curve

$$\mathcal{C} = \phi(\mathbf{P}^1(\mathbf{R})).$$

The value of $\phi(\alpha_i)$ is interpreted in $\mathbf{P}^1(\mathbf{R}^n)$, i.e., it is an asymptotic line.

Let

$$\boldsymbol{b}_i = \frac{1}{n}(-1, \ldots, -1, n-1, -1, \ldots, -1), \quad (i = 1, 2, \ldots, n).$$

and set

$$\boldsymbol{c}_i = \boldsymbol{b}_i + \frac{1}{n-1}\boldsymbol{b}_n, \quad (i < n) \quad (\boldsymbol{c}_i \perp \boldsymbol{b}_n)$$

Then,

$$\begin{aligned} \phi(t) &= \sum_{i=1}^{n} \log \frac{|t - \alpha_i|}{|f'(\alpha_i)|^{1/(n-2)}} \cdot \boldsymbol{b}_i \\ &= \sum_{i=1}^{n-1} \log \frac{|t - \alpha_i|}{|f'(\alpha_i)|^{1/(n-2)}} \cdot \boldsymbol{c}_i + \ell_n \cdot \sqrt{\frac{n}{n-1}}\,\boldsymbol{b}_n, \end{aligned}$$

where

$$\ell_n = \frac{\log \dfrac{|t - \alpha_n|}{|f'(\alpha_n)|^{1/(n-2)}} - \dfrac{1}{n-1} \displaystyle\sum_{i=1}^{n-1} \log \frac{|t - \alpha_i|}{|f'(\alpha_i)|^{1/(n-2)}}}{\sqrt{n/(n-1)}}.$$

Therefore, $\phi(t)$ approaches the line

$$\mathcal{L}_n = \sum_{i=1}^{n-1} \log \frac{|\alpha_n - \alpha_i|}{|f'(\alpha_i)|^{1/(n-2)}} \cdot \boldsymbol{c}_i + \mathbf{R}\boldsymbol{b}_n$$

as t approaches α_n. If $t = \alpha_n + u$ with $|u|/(\alpha_n - \alpha_{n-1}) \ll 1$, we have

$$\begin{aligned} \operatorname{dist}(\phi(t), \mathcal{L}_n) &= \left\| \sum_{i=1}^{n-1} \log \frac{|t-\alpha_i|}{|\alpha_n - \alpha_i|} \cdot \boldsymbol{c}_i \right\| \ll \frac{(n-1)|u|}{\alpha_n - \alpha_{n-1}}; \qquad (3) \\ \ell_n &= \frac{\log |u|}{\sqrt{n/(n-1)}} + O_{\mathcal{A}}(1). \end{aligned}$$

If we write $r = \|\phi(t)\|$, then the two estimates implies $r = -\ell_n + O_{\mathcal{A}}(1)$. By substituting this in (3), we get

$$\operatorname{dist}(\phi(t), \mathcal{L}_n) \ll_{\mathcal{A}} \exp\left(-\sqrt{\frac{n}{n-1}}\, r \right). \qquad (4)$$

The transcendental curve $\mathcal{C}$ has convexity in a certain sense. We need a linear projection for observing it. For a suitable constant vector $\boldsymbol{v}$, we have.

$$\begin{aligned} \phi(t) - \boldsymbol{v} &= \sum_{i \neq 2} \log|t - \alpha_i| \cdot \boldsymbol{c}_i + \ell_2 \cdot \sqrt{\tfrac{n}{n-1}}\, \boldsymbol{b}_2 \\ &= \sum_{i \neq 2,4} \log \frac{|t-\alpha_i|}{|t-\alpha_4|} \cdot \boldsymbol{c}_i + \ell_2 \cdot \sqrt{\tfrac{n}{n-1}}\, \boldsymbol{b}_2. \end{aligned}$$

Since $\boldsymbol{c}_1, \boldsymbol{b}_2, \boldsymbol{c}_3, \boldsymbol{c}_5, \boldsymbol{c}_6 \ldots, \boldsymbol{c}_n$ form a basis of the orthogonal space $\Pi_{\log}$ of $(1, 1, \ldots, 1)$,

$$(u(t), w(t)) = \left(\log \frac{|t-\alpha_1|}{|t-\alpha_4|}, \log \frac{|t-\alpha_3|}{|t-\alpha_4|} \right)$$

is a linear projection of $\phi(t)$.

The curve $(u(t), w(t))$ with $t \in]\alpha_1, \alpha_3[$ is a convex curve. To verify this, observe

$$\frac{du}{dt} = \frac{\alpha_1 - \alpha_4}{(t-\alpha_1)(t-\alpha_4)} > 0 > \frac{\alpha_3 - \alpha_4}{(t-\alpha_3)(t-\alpha_4)} = \frac{dw}{dt}$$

and calculate

$$\frac{d^2 w}{du^2} = \frac{\frac{d}{dt}\frac{dw/dt}{du/dt}}{du/dt} = \frac{(t-\alpha_1)(t-\alpha_4)}{\alpha_3-\alpha_4}\,\frac{d}{dt}\,\frac{t-\alpha_3}{t-\alpha_1}$$
$$= -\frac{(\alpha_1-\alpha_3)(t-\alpha_4)}{(\alpha_3-\alpha_4)(t-\alpha_1)} > 0.$$

Since we can transform α_{m-2}, α_{m-1} and α_m to $+1$, -1 and 0, the "convexity" is also valid for other intervals of the form $]\alpha_{m-1}, \alpha_{m+1}[$. (Subscripts are read cyclically.) The "convexity" implies the following:

Lemma 2 *Let $m \in \{1, 2, \ldots, n\}$. No three points of the part*

$$\phi\left(]\alpha_{m-1}, \alpha_{m+1}[\right)$$

are collinear.

We can also prove a similar result by directly working with ϕ.

Lemma 3 *A hyperplane of $\Pi_{\log}$ intersects $\mathcal{C}$ at no more than $2n-2$ points.*

To see this, we write $(w_1, w_2, \ldots, w_n) \in \Pi_{\log}$ for the normal vector of the hyperplane and count solutions to

$$c = \sum_{i=1}^{n} w_i \phi_i(t) = \sum_{i=1}^{n} w_i \log \left| \frac{D^{1/n(n-2)}\,(x - y\alpha_m)}{|f'(\alpha_m)|^{1/(n-2)}} \right|.$$

By absorbing constant terms in c', we get

$$c' = \sum_{i=1}^{n} w_i \log \left| \frac{t-\alpha_i}{|f(t)|^{1/n}} \right| = \sum_{i=1}^{n} w_i \log |t-\alpha_i|\,.$$

We easily differentiate the right hand side

$$\frac{d}{dt}\sum_{i=1}^{n} w_i \log|t-\alpha_i| = \frac{\sum_{i=1}^{n} w_i f_i(t)}{f(t)},$$

where $f_i(t) = f(t)/(t-\alpha_i)$ is a monic of degree $\leq n-1$. Thus, the function $\sum_{i=1}^{n} w_i \log|t-\alpha_i|$ has at most $n-2$ critical points. Hence, its mapping degree is at most $2n-2$. Now, the Lemma is established. **qed.**

4 Admissible Transformation

Let $G \in GL_2(\mathbf{R})$. We say G is admissible for $\mathcal{A}$ if it preserves

$$\left|\det\begin{pmatrix} x & x' \\ y & y' \end{pmatrix}\right|$$

and is compatible with change of variables:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \pm G \begin{pmatrix} x \\ y \end{pmatrix}.$$

These conditions can be thought of as preservation of discreteness.

The former condition is equivalent to $|\det G| = 1$ under the latter condition. The latter is characterized by (1). Therefore, admissibility is characterized by

$$|\det G| = \left|\prod_{i=1}^{n} \chi(G, \alpha_i)\right| = 1. \tag{5}$$

An admissible transformation preserves discriminant:

$$D(\tilde{\mathcal{A}}) = D(\mathcal{A})$$

since

$$\prod_{i \neq j}^{n} |\tilde{\alpha}_i - \tilde{\alpha}_j| = \prod_{i \neq j}^{n} \left| \frac{(\alpha_i - \alpha_j)\det G}{\chi(G, \alpha_i)\chi(G, \alpha_j)} \right|.$$

Admissible transformation has freedom of degree 2, i.e., it can transform given two points, say μ and ν, (other than those from $\mathcal{A}$) to 0 and ∞. To do this, choose v and w suitably to make

$$G = \begin{pmatrix} v & -v\mu \\ w & -w\nu \end{pmatrix}$$

satisfy (5).

5 Arithmetic Point

We say $f(X, Y)$ is arithmetic ($\mathcal{A}$ is arithmetic) if $f(X, Y) \in \mathbf{Z}[X, Y]$ is an irreducible monic polynomial. We say t is arithmetic if $f(X, Y)$ is arithmetic and $x(t), y(t) \in \mathbf{Z}$. We also say $\phi(t)$ is arithmetic if t is arithmetic.

We extend this notion of arithmeticity by admissible transformation: assume $G \in GL_2(\mathbf{R})$ is admissible for $\mathcal{A}$; then $\tilde{f}(X,Y)$ is arithmetic if $f(X,Y)$ is arithmetic; and $\tilde{t}$ is arithmetic if t is arithmetic. This way of extension is compatible with arithmeticity of $\phi(t)$ since $\phi(t)$ is invariant under G.

When t and t' are arithmetic, $\phi(t) - \phi(t')$ is an image of a regulator map:

$$\phi(t') - \phi(t) = \log \vec{\varepsilon} = \left(\log \left|\varepsilon^{(i)}\right|\right)_{1 \le i \le n} \in \mathfrak{E},$$

where $\varepsilon^{(i)} = (x' - y'\alpha_i)/(x - y\alpha_i)$. This is easily verified by (2).

Pohst's lower bound

$$\|\log \varepsilon\|^2 \ge n \log^2 \frac{1+\sqrt{5}}{2}$$

for $\log \varepsilon \in \mathfrak{E} \setminus \{0\}$ guarantees constant distance between arithmetic points $\phi(t)$ and $\phi(t')$. However, its power will become more important when units outside $\mathcal{C}$ are involved. Indeed, we have

Lemma 4 *Let $\phi(t)$, $\phi(t')$ and $\phi(t'')$ be arithmetic points. Assume they are not collinear. Denote by S the area of the triangle formed by them. Then, we have*

$$S \ge \frac{\sqrt{3}}{4} n^2 \log^2 \frac{1+\sqrt{5}}{2}.$$

This is shown by lattice reduction and Pohst's lower bound: reduced basis of the lattice $\mathbf{Z}(\phi(t') - \phi(t)) + \mathbf{Z}(\phi(t'') - \phi(t))$ spans a triangle whose area is larger than the right hand side; obviously the area of this triangles equals S. **qed.**

6 Exponential Gap Principle in Degree n

Lemma 5 *Assume arithmetic points $t, t', t'' \in \,]\alpha_{m-1}, \alpha_m[\, \cup \,]\alpha_m, \alpha_{m+1}[$ are in a sufficiently small neighbourhood of α_m whose radius depends on $\mathcal{A}$. Assume also $r'' = \|\phi(t'')\| \ge r' = \|\phi(t')\| \ge r = \|\phi(t)\|$. Then, we have*

$$r'' \gg_{\mathcal{A}} \exp\left(\sqrt{\frac{n}{n-1}}\, r\right).$$

This is shown as follows. When $|t-\alpha_m| \ll_{\mathcal{A}} 1$, the inequality (4) implies

$$S \ll_{\mathcal{A}} \|\phi(t'') - \phi(t)\| \exp\left(-\sqrt{\frac{n}{n-1}}\, r\right),$$

where S denotes the area of the triangle formed by $\phi(t), \phi(t')$ and $\phi(t'')$. Lemma 4 now establishes the Lemma. **qed.**

7 Transcendental Curve $\mathcal{C}$ in Quartic Case

Hereafter, we assume $n = 4$. We write $\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3$ and $\delta = \alpha_4$. Recall

$$\begin{aligned} \boldsymbol{b}_1 &= -\tfrac{1}{4}(-3,1,1,1), & \boldsymbol{b}_2 &= -\tfrac{1}{4}(1,-3,1,1), \\ \boldsymbol{b}_3 &= -\tfrac{1}{4}(1,1,-3,1), & \boldsymbol{b}_4 &= -\tfrac{1}{4}(1,1,1,-3). \end{aligned}$$

and set

$$\boldsymbol{e}_i = \boldsymbol{b}_i + \boldsymbol{b}_4, \quad (i = 1,2,3).$$

Then, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ form a orthonormal basis of the space $\Pi_{\log}$. The vectors $\boldsymbol{b}_i$'s are easily written in terms of the orthonormal basis.

$$\begin{aligned} 2\boldsymbol{b}_1 &= +\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3, & 2\boldsymbol{b}_2 &= -\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3, \\ 2\boldsymbol{b}_3 &= -\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3, & 2\boldsymbol{b}_4 &= +\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3. \end{aligned}$$

Thus, orthonormal coordinates $z_1(t), z_2(t)$ and $z_3(t)$ of $\phi(t)$ are calculated as follows.

$$z_1(t)\boldsymbol{e}_1 + z_2(t)\boldsymbol{e}_2 + z_3(t)\boldsymbol{e}_3 = \phi(t) = \frac{1}{2}\sum_{i=1}^{4} \log \frac{|t-\alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot 2\boldsymbol{b}_i$$

$$= \frac{1}{2}\log\left|\frac{(t-\alpha)(t-\delta)(\gamma-\beta)}{(t-\beta)(t-\gamma)(\delta-\alpha)}\right| \cdot \boldsymbol{e}_1 + \frac{1}{2}\log\left|\frac{(t-\beta)(t-\delta)(\gamma-\alpha)}{(t-\alpha)(t-\gamma)(\delta-\beta)}\right| \cdot \boldsymbol{e}_2$$

$$+\frac{1}{2}\log\left|\frac{(t-\gamma)(t-\delta)(\beta-\alpha)}{(t-\alpha)(t-\beta)(\delta-\gamma)}\right| \cdot \boldsymbol{e}_3$$

The argument for Lemma 3 implies $z_i(t)$ has at most 2 critical points. Therefore, $z_1(t)$ has exactly one critical point in each of $]\beta,\gamma[$ and $]\delta,\alpha[$. We call them $\mu(\beta,\gamma)$ and $\mu(\delta,\alpha)$. Similarly, $\mu(\alpha,\beta)$ and $\mu(\gamma,\delta)$ are defined to be the critical points of $z_3(t)$ in the indicated interval.

We transform $\{\mu(\beta,\gamma),\mu(\delta,\alpha)\}$ to $\{0,\infty\}$ by an admissible transformation. and assume

$$\alpha = -\delta, \quad \beta = -\gamma.$$

The cross ratio

$$\lambda = -\frac{(\gamma-\beta)(\alpha-\delta)}{(\delta-\gamma)(\beta-\alpha)}$$

of the configuration $\mathcal{A}$ is a projective invariant (upto permutation of "roots"). It actually is known to characterize projective configuration.

The admissible transformation determined by $\{\mu(\alpha,\beta),\mu(\gamma,\delta)\} \longmapsto \{0,\infty\}$ inverts λ. Using this, we assume $4\gamma\delta/(\delta-\gamma)^2 = \lambda \geq 1$ without loss of generality. We will refer to this kind of configurations by normalized configurations.

Note: The configuration is called harmonic when its cross ratio equals 1. This happens in Diophantine situation. Let $\delta = (\sqrt{2}+1)\sqrt{6}$, $\gamma = (\sqrt{2}-1)\sqrt{6}$, $\beta = -\gamma$ and $\alpha = -\delta$. Then, we have $\lambda = 1$ and $f(X,Y) = x^4 - 36x^2y^2 + 36y^4$. There are non-trivial arithmetic points $f(1,1) = f(1,-1) = 1$.

We have $\gamma \geq \delta/(3+2\sqrt{2})$ since $\lambda \geq 1$. Set $L = \gamma + \delta$. Then,

$$\sqrt{D} = 4\gamma\delta L^2(\delta-\gamma)^2 \leq L^6/\lambda. \tag{6}$$

We now have

$$2z_1(t)\boldsymbol{e}_1 + 2z_2(t)\boldsymbol{e}_2 + 2z_3(t)\boldsymbol{e}_3 = 2\phi(t)$$

$$= \log\left|\frac{2\gamma(t-\alpha)(t-\delta)}{2\delta(t-\beta)(t-\gamma)}\right|\boldsymbol{e}_1 + \log\left|\frac{(t-\beta)(t-\delta)}{(t-\alpha)(t-\gamma)}\right|\boldsymbol{e}_2 + \log\left|\frac{(t-\gamma)(t-\delta)}{(t-\alpha)(t-\beta)}\right|\boldsymbol{e}_3.$$

We set $\mu = -\mu(\alpha,\beta) = \mu(\gamma,\delta) = \sqrt{\gamma\delta}$. Then, the curve $\mathcal{C}$ is preserved by $t \longmapsto -t$, $t \longmapsto -\mu^2/t$ and $t \longmapsto \mu^2/t$. These transformations expressed by

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} & -\mu \\ \mu^{-1} & \end{pmatrix}, \quad \begin{pmatrix} & \mu \\ \mu^{-1} & \end{pmatrix}$$

are admissible for $\mathcal{A}$. In $\Pi_{\log}$, the three transformations are rotations of order 2 around $\mathbf{R}\boldsymbol{e}_1$, $\mathbf{R}\boldsymbol{e}_2$ and $\mathbf{R}\boldsymbol{e}_3$.

We wrap $\mathcal{C}$ by 5 cylinders (4 "asymptotic cylinders" and "the bridge"). The part of $\mathcal{C}$ corresponding to $\phi(\delta+u)$ with

$$s^{-1} := \frac{|u|}{\delta-\gamma+u} \leq \frac{4}{L^2} \tag{7}$$

will be called the asymptotic part of $\mathcal{C}$ at δ. Asymptotic part of $\mathcal{C}$ at other "roots" are defined by symmetry. The rest of the part of $\mathcal{C}$ will be called the bridge.

We wrap the bridge by *the parallelotope $\mathcal{B}$ of bridge.*

$$\mathcal{B} = \left\{ v_1 \boldsymbol{e}_1 + v_2 \boldsymbol{e}_2 + v_3 \boldsymbol{e}_3 : |v_1| < \log(L/2), |v_2| < C, |v_3| < \log L\sqrt{\lambda} \right\},$$

where C is a suitable number close to $\log(L/2)$. Calculation shows $\phi(\delta+u) \in \mathcal{B}$ provided

$$\delta + u \geq \mu$$

and

$$s^{-1} = \frac{|u|}{\delta - \gamma + u} \geq \frac{4}{L^2}.$$

By symmetry, we verify $\mathcal{B}$ actually contains the bridge.

In the asymptotic part $s^{-1} = |u|/(\delta - \gamma + u) \leq 4/L^2$ at δ, we have

$$\mathrm{dist}(\phi(\delta + u), \mathcal{L}_4) \ll s^{-1}$$

by (3). Therefore, the asymptotic part is contained in an infinite cylinder of radius proportional to $1/L^2$.

We need more information on approach of $\phi(t)$ to the asymptotic line. Calculating $\phi(\delta + u)$, we see

$$\begin{aligned} (2r)^2 &> 2(\log s)^2 + (\log s + \log \lambda - 0.2)^2, \\ (2r)^2 &< 2(\log s + 2)^2 + (\log s + \log \lambda + 2)^2. \end{aligned}$$

Here, we have $\lambda < s^3$ since $1 \leq \lambda \leq L^6/\sqrt{D}$. Thus, we get

$$\sqrt{2}\, r/3 + 2 < \log s < 2\, r/\sqrt{3} + 0.1 \tag{8}$$

and

$$\mathrm{dist}(\phi(\delta + u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}. \tag{9}$$

Similarly estimate hold for the other asymptotic parts.

8 Arithmetic Points in Quartic Case

Hereafter, we assume $f(X, Y)$ is arithmetic. We will find nice gap principles. We also count arithmetic points on the bridge. Counting of arithmetic points in the asymptotic part is postponed to the next section, where we appeal to Baker theory.

Firstly, dependence the implied constant of Lemma 5 can be made explicit as in the following:

Lemma 6 *Let*

$$t, t', t'' \in]\mu(\alpha_{m-1}, \alpha_m), \alpha_m[\cup]\alpha_m, \mu(\alpha_m, \alpha_{m+1})[$$

be arithmetic points whose image under ϕ are in the asymptotic part at α_m. Assume also $r'' = \|\phi(t'')\| \geq r' = \|\phi(t')\| \geq r = \|\phi(t)\|$. Then, we have

$$r'' \gg e^{\sqrt{2}\, r/3}.$$

Assume $\mathcal{A}$ is normalized and two points t and t' are arithmetic. Write

$$\varepsilon(\alpha_i) = \frac{x' - y'\alpha_i}{x - y\alpha_i}.$$

Then, we have

$$\begin{aligned} |\varepsilon(\alpha) - \varepsilon(\gamma)| &= \frac{L \left| \det \begin{pmatrix} y' & x' \\ y & x \end{pmatrix} \right|}{|(x - y\alpha)(x - y\gamma)|} \geq \frac{L}{|(x - y\alpha)(x - y\gamma)|}, \\ |\varepsilon(\beta) - \varepsilon(\delta)| &\geq \frac{L}{|(x - y\beta)(x - y\delta)|}. \end{aligned}$$

Therefore, we get

$$|\varepsilon(\alpha) - \varepsilon(\gamma)| \ |\varepsilon(\beta) - \varepsilon(\delta)| \geq L^2$$

and hence

$$\max\{|\varepsilon(\alpha)|, |\varepsilon(\gamma)|\} \ \max\{|\varepsilon(\beta)|, |\varepsilon(\delta)|\} \geq (L/2)^2.$$

Noting that $\varepsilon(\alpha)\varepsilon(\beta)\varepsilon(\gamma)\varepsilon(\delta) = \pm 1$, we also get

$$\min\{|\varepsilon(\alpha)|, |\varepsilon(\gamma)|\} \ \min\{|\varepsilon(\beta)|, |\varepsilon(\delta)|\} \leq (L/2)^{-2}.$$

Thus, we have either

$$|z_3(t') - z_3(t)| = \frac{1}{2}\left|\log\frac{|\varepsilon(\alpha)\varepsilon(\beta)|}{|\varepsilon(\gamma)\varepsilon(\delta)|}\right| \geq 2\log\frac{L}{2}$$

or

$$|z_1(t') - z_1(t)| = \frac{1}{2}\left|\log\frac{|\varepsilon(\alpha)\varepsilon(\delta)|}{|\varepsilon(\beta)\varepsilon(\gamma)|}\right| \geq 2\log\frac{L}{2}.$$

Assume further $\phi(t), \phi(t') \in \mathcal{B}$. Then, the former holds since the latter is impossible by the definition of $\mathcal{B}$.

Noting that the longest side of $\mathcal{B}$ is estimated by

$$2\log L\sqrt{\lambda} < 8\log L - \log D < 8\log(L/2),$$

we see

Lemma 7 *There are at most* 4 *arithmetic points in* $\mathcal{B}$*.*

9 Logarithmic Forms

Set $\mathfrak{K} = \mathbf{Q}(\alpha_1)$. The following Lemma is the last of algebraic assertions and is used for controlling a lower bound of logarithmic form.

Lemma 8 *Let R be a positive parameter and assume that there are* 7 *arithmetic points $\phi(t)$ such that $r := \|\phi(t)\| \leq R$. Then, $\mathfrak{O}(\mathfrak{K})^\times$ is generated by a triple (ζ, η, ξ) such that*

$$\|\zeta\| \leq \|\eta\| \leq \|\xi\| \leq 2R.$$

Let $\gamma \in (\mathfrak{K}) \setminus \{0\}$. Then, γ is associated with some $\gamma' \in (\mathfrak{K})$ such that

$$\|\log\vec{\gamma}\,'\| \leq 2R + \log N_{\mathfrak{K}/\mathbf{Q}}\gamma\,'.$$

Indeed, Lemma 3 implies that differences of the form $\phi(t') - \phi(t)$ with $\|\phi(t)\|, \|\phi(t')\| \leq R$ generate a sublattice of rank 3 of $\mathcal{E} = \log\mathfrak{O}(\mathfrak{K})$. Therefore, reduction of Seeber-Dirichlet implies the first assertion.

Choose suitable associate γ' of γ such that $\log\vec{\gamma}\,' - (1/n)\log(N_{\mathfrak{K}/\mathbf{Q}}\gamma\,'\cdot\vec{1})$ belongs to the Voronoï domain of the origin with respect to $\log\mathfrak{E}$. We have, $\|\log\vec{\gamma}\,' - (1/n)\log(N_{\mathfrak{K}/\mathbf{Q}}\gamma\,'\cdot\vec{1})\| \leq 2R$. Thus, the Lemma is established. **qed.**

Let $t = \delta + u$ and assume $\phi(t)$ belongs to the asymptotic part at δ. We have seen

$$\operatorname{dist}(\phi(\delta+u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}$$

when $\mathcal{A}$ is normalized. The left hand side has an invariant representation:

$$9 \quad \cdot \quad \operatorname{dist}(\phi(\delta+u), \mathcal{L}_4)^2 = \log^2\left|\frac{(t-\alpha)(\delta-\beta)}{(\delta-\alpha)(t-\beta)}\right|$$
$$+\log^2\left|\frac{(t-\beta)(\delta-\gamma)}{(\delta-\beta)(t-\gamma)}\right| + \log^2\left|\frac{(t-\gamma)(\delta-\alpha)}{(\delta-\gamma)(t-\alpha)}\right|. \qquad (10)$$

We get the inequality

$$\Lambda := \log\left|\frac{(t-\alpha)(\delta-\beta)}{(\delta-\alpha)(t-\beta)}\right| \ll e^{-\sqrt{2}r/3},$$

of invariant quantities under $GL_2(\mathbf{R})$.

Hereafter, we assume $f(X, Y)$ is an irreducible quartic polynomial in $\mathbf{Z}[X, Y]$. A lower bound of logarithmic form is of the form

$$\log|\Lambda| > -\log B \cdot \prod_{k=1}^{4} A_k \cdot \log\left(\prod_{k=1}^{4} A_k\right),$$

where A_k's are heights of fundamental units of $Q(\alpha)$ ($k = 2, 3, 4$) and reduction Δ of $(\delta-\beta)/(\delta-\alpha)$ module suitable units ($k = 1$); and B is the maximum of the (integer) coefficients of the linear form

$$\log\left|\frac{(t-\alpha)(\delta-\beta)}{(\delta-\alpha)(t-\beta)}\right| = m_1 \log|\zeta'| + m_2 \log|\eta'| + m_3 \log|\xi'| + m_4 \log|\Delta|,$$

where ζ', η' and ξ are generators of the group generated by $\zeta^{1-\sigma}, \eta^{1-\sigma}, \xi^{1-\sigma}$ for some conjugation that carries α to β. (When the four logarithms in the right hand side are linearly dependent over $\mathbf{Q}$, we can set several A_j's to be 1, remove corresponding logarithms.) We assume the image of ζ', η' and ξ' under the Dirichlet regulator map form a reduced basis of a suitable lattice. The suitable units for reduce $(\delta-\beta)/(\delta-\alpha)$ is the group generated by ζ', η' and ξ'.

The parameters are estimated by the following Lemmata:

Lemma 9 *Let $\phi(t)$ be an arithmetic point. Then, $A_1 \ll \|\phi(t)\| + \log D$.*

Lemma 10 *Let R be a positive parameter and assume that there are 7 arithmetic points $\phi(t)$ such that $r := \|\phi(t)\| \leq R$. Then, $A_i \ll R$ for $i = 2, \cdots, 4$.*

Lemma 11 *Let $\phi(t)$ be an arithmetic point. Then, $B \ll \|\phi(t)\| + \log D$.*

Proof of Lemma 9. We have $\mathrm{dist}(\phi(t), \mathcal{L}_i) \leq \|\phi(t)\| + \mathrm{dist}(0, \mathcal{L}_i)$ and $\mathrm{dist}(0, \mathcal{L}_i) \ll \log \lambda \ll \log D$. Thus, $\mathrm{dist}(\phi(t), \mathcal{L}_i) \ll \|\phi(t)\| + \log D$.

By (10), this implies

$$\log \left| \frac{(t-\alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t - \alpha_k)} \right| \ll \|\phi(t)\| + \log D$$

for every triple (i, j, k) of distinct indices. Therefore, contribution of infinite places to the height is is $O(\|\phi(t)\| + \log D)$.

On the other hand, contribution of finite places to the height is $O(\log D)$ since the norm of the algebraic integer $\alpha_i - \alpha_j$ to $\mathbf{Q}$ divides D.

The Lemma now follows by noting that $(t-\alpha)/(t-\beta) = (x-y\alpha)/(x-y\beta)$ is an element of the group of units suitable for reduction. **qed.**

Proof of Lemma 10. The Lemma is immediate after Lemma 8. **qed.**

Proof of Lemma 11. Let ψ be the composition of the component-wise logarithm and Minkowski embedding. Then, we have $\|\psi(\Delta)\| \ll \|\phi(t)\| + \log D$ by Lemma 9. Thus, we get $\|\psi((\zeta')^{m_1}(\eta')^{m_2}(\xi')^{m_3})\| \ll \|\phi(t)\| + \log D$. Since ζ', η' and ξ' gives a reduced basis, this implies $m_1\|\psi(\zeta')\|$, $m_2\|\psi(\eta')\|$, $m_3\|\psi(\xi')\| \ll \|\phi(t)\| + \log D$. Therefore, we get $m_1, m_2, m_3 \ll \|\phi(t)\| + \log D$. (N.B. Recall Pohst lower bound.)

The Lemma now follows by noting $m_4 = 1$. **qed.**

10 Conclusion

Proof of Theorem 1. *Assume $\mathfrak{K}$ is normal. Suppose 15 arithmetic points exist for contradiction.* Remove 6 arithmetic points of minimal "radii" $\|\phi(t)\|$. Then, the arithmetic points on the bridge are removed by Lemma 7. For the rest of the arithmetic points t, $r = \|\phi(t)\|$ is suitable for R in Lemma 10. Therefore, we get $A_k \ll r + \log D$ $(k = 2, 3, 4)$. Here, we have $\log D \ll r$ by inequalities (6), (7) and (8). Thus, we get $A_k \ll r$. Lemma 9 implies $A_1 \ll r$.

Since the remaining 9 arithmetic points are distributed in the 4 asymptotic parts, at least 3 arithmetic points t, t' and t'' concentrate on one asymptotic part. Write $r'' = \|\phi(t'')\|, r' = \|\phi(t')\|, r = \|\phi(t)\|$.

$$\begin{aligned} r'' &\ll \log B(\phi(t'')) \cdot \prod_{k=1}^{4} A_k \cdot \log\left(\prod_{k=1}^{4} A_k\right) \\ &\ll \log r'' \cdot r^4 \cdot \log r, \end{aligned}$$

where we used $B(\phi(t'')) \ll r$ that follows from Lemma 11.

Thus,

$$r'' \ll r^4 \cdot \log^2 r$$

Using Lemma 6, we get

$$e^{\sqrt{2}r/3} \ll r^4 \cdot \log^2 r$$

We now conclude

$$\log D \ll r \ll 1.$$

qed.

Ryotaro Okazaki
Doshisha University,
Dept. Mathematics
Kyotanabe-shi, Kyoto-fu,
610-0394 JAPAN
E-mail: rokazaki@dd.iij4u.or.jp