Does population aging promote economic growth?

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Abstract

It is one of the controversial issues whether population aging promotes economic growth because it affects public attitudes to population aging and the related policies. In this paper we answer the question in a simple growth model with altruistically linked overlapping generations.

We have three analytical results. First, population aging is neutral to economic growth when an adult mortality rate is relatively high and bequests are operative. Second, if the mortality rate falls beyond a critical level, population aging promotes economic growth because of the over-investment of physical capital in a bequest constrained economy. Third, population aging is more likely to promote economic growth if the altruism is smaller and/or the role of education time in human capital formation is more important.

Our result suggests a rationale of a pay-as-you-go public pension that is neutral to the allocation of time to alleviate the bequest constraint.

JEL Classification J14, J24, O41

Keywords Population aging, Growth, Altruism, Bequests, Human capital

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1 Introduction

It is one of the controversial issues in population economics whether population aging promotes economic growth and welfare because it affects public attitudes to population aging and the related policies. In this paper we answer the question in a simple model of endogenous growth with altruistically linked overlapping generations.

We have three analytical results. First, population aging is neutral to economic growth when an adult mortality rate is relatively high and bequests are operative. Second, if the mortality rate falls beyond a critical level, population aging promotes economic growth because of the overinvestment of physical capital in a bequest constrained economy. Third, population aging is more likely to promote economic growth if the altruism is smaller and/or the role of education time in human capital formation is more important.

This paper contributes to especially two fields of research. First, it presents a new insight into a discussion on the growth effect of population aging.[1] A degree of freedom of intergenerational transfers matters. If bequests are operative, population aging is neutral to economic growth because of the neutrality à la Barro (1974). Otherwise, it has a positive growth effect because of the overinvestment of physical capital in a bequest constrained economy (Weil (1987), Caballé (1995)).

Second, some papers examine necessary and sufficient conditions for the bequest constraint to be binding (Drazen (1978), Abel (1987), Weil (1987), Caballé (1995), Cardia and Michel (2004)). One of the common result is that the bequest constraint becomes binding when an inter-cohort discount factor is small, i.e., parents do not care about their children so much. This paper adds a new insight to this field of research. In a process of population aging, the bequest constraint is more likely to be binding because parents care about their own future life rather than their children. If it is the case, we may observe that people look like selfish in aging economies, although their preferences do not change.

The remainder of the paper is structured as follows. Section 2 sets up the basic model and analyzes the equilibrium and dynamics when bequests are operative. Section 3 analyzes a case in which bequests are not operative. Section 4 combines the results in Section 2 and 3 to trace the equilibrium path in a process of population aging. The final section concludes the paper.

2 Model

We use a three-period overlapping generations model with an adult mortality risk. Population newly born is constant and normalized to unity.[2] In the first period of life, an individual born in period $(t-1)$ receives education from his parent in terms of time l_{t-1}, and goods e_{t-1}. This education forms human

[1] Recently some papers show population aging may prevent economic growth (Zhang, Zhang, and Lee (2003), van Groezen, Meijdam, and Verbon (2005), Miyazawa (2006)), although relatively large number of papers conclude population aging promotes economic growth (Ehrlich and Lui (1991), Pecchenino and Pollard (1999), Futagami and Nakajima (2001), Zhang and Zhang (2001), Zhang, Zhang, and Lee (2001)).

[2] This assumption is not essential and we use it to focus on the pure effect of population aging.

capital in the second period h_t. In the second period, he receives a bequest from his parent $(1+r_t)b_t$, allocates one unit of time between education for his child l_t, and working $1-l_t$, transfers to his child education goods e_t, consumes c_{1t}, and purchases annuities s_t for his retirement. At the beginning of the third period, he faces a mortality risk. He survives into the third period with probability $0 \leq q \leq 1$, or dies with probability $1-q$. In the third retirement period, he receives the return of annuities to consume c_{2t+1}.

The budget constraints in the second and third period are respectively given by

$$\begin{aligned} (1+r_t)b_t + w_t h_t(1-l_t) &= c_{1t} + e_t + s_t + b_{t+1} && (1) \\ R_{t+1}s_t &= c_{2t+1} && (2) \end{aligned}$$

where w_t and r_t stand for a wage rate per effective labor and an interest rate, respectively. R_{t+1} stands for the rate of return of actuarially fair annuities. Equations (1) and (2) imply that an individual is assumed to use annuities to finance his own consumption in the retirement and risk-free assets to finance transfers to his child[3].

The technology of human capital formation is specified by a Cobb-Douglas form,

$$h_{t+1} = Be_t^{1-\gamma}(h_t l_t)^{\gamma} \quad (3)$$

where $0 \leq \gamma \leq 1$ is an elasticity of the effective time input, and $B > 0$ is a productivity parameter.

The household maximization problem is formulated as

$$V(b_t, h_t) = \max \quad \ln c_{1t} + \beta q \ln c_{2t+1} + \rho V(b_{t+1}, h_{t+1})$$

subject to equations (1), (2), and (3). $V(b_{t+1}, h_{t+1})$ stands for the indirect utility of his child. $0 < \beta < 1$ is a subjective discount factor and $0 < \rho < 1$ is an inter-cohort discount factor.

The optimality conditions require

$$\begin{aligned} \frac{1}{c_{1t}} - \lambda_t &= 0 && (4) \\ \frac{\beta q}{c_{2t+1}} - \frac{\lambda_t}{R_{t+1}} &= 0 && (5) \\ \rho \frac{\partial V_{t+1}}{\partial b_{t+1}} - \lambda_t &\leq 0 && (6) \\ \rho \frac{\partial V_{t+1}}{\partial h_{t+1}} \frac{\partial h_{t+1}}{\partial e_t} - \lambda_t &= 0 && (7) \\ \rho \frac{\partial V_{t+1}}{\partial h_{t+1}} \frac{\partial h_{t+1}}{\partial l_t} - \lambda_t w_t h_t &= 0 && (8) \end{aligned}$$

where λ_t stands for a multiplier attached to the lifetime budget constraint. Equation (6) holds with equality when bequests are operative, which is assumed in this section. A bequest-constraint economy is examined in the next section.

Equations (1)-(8) solve for $c_{1t}, c_{2t+1}, s_t, b_{t+1}, e_t, l_t, h_{t+1}, \lambda_t$ as functions of b_t and h_t.

[3] This portfolio choice is optimal if the rate of return of annuities is actuarially fair. See Fuster (2000).

From the envelope theorem, we have

$$\frac{\partial V_t}{\partial b_t} = \lambda_t(1+r_t) \tag{9}$$
$$\frac{\partial V_t}{\partial h_t} = \lambda_t w_t \tag{10}$$

From equations (6) and (9),

$$\frac{\lambda_t}{\lambda_{t+1}} = \rho(1+r_{t+1}) \tag{11}$$

Equation (11) gives the rate of economic growth on a balanced growth path as shown below.

Substituting equation (10) into equations (7) and (8), we have

$$\lambda_t e_t = \rho(1-\gamma)\lambda_{t+1}w_{t+1}h_{t+1} \tag{12}$$
$$\lambda_t w_t h_t l_t = \rho\gamma\lambda_{t+1}w_{t+1}h_{t+1} \tag{13}$$

Equations (12) and (13) gives the total cost of education such as

$$e_t + w_t h_t l_t = \frac{\rho\lambda_{t+1}w_{t+1}h_{t+1}}{\lambda_t} \tag{14}$$

Substituting equations (4), (5), (11), and (14) into the lifetime budget constraint, we have

$$\lambda_t[(1+r_t)b_t + w_t h_t] = 1 + \beta q + \rho\lambda_{t+1}[(1+r_{t+1})b_{t+1} + w_{t+1}h_{t+1}]$$

which regulates the law of motion of $\lambda_t[(1+r_t)b_t + w_t h_t]$. Since $0 < \rho < 1$, we have a unique solution of

$$\lambda_t[(1+r_t)b_t + w_t h_t] = \frac{1+\beta q}{1-\rho} \tag{15}$$

Substituting equation (15) into equations (4) and (5), we have

$$c_{1t} = \frac{1-\rho}{1+\beta q}[(1+r_t)b_t + w_t h_t] \tag{16}$$
$$c_{2t+1} = \frac{\beta q(1-\rho)}{1+\beta q}[(1+r_t)b_t + w_t h_t]R_{t+1} \tag{17}$$

Substituting equation (11) into equations (12) and (13), we have

$$e_t = \frac{(1-\gamma)w_{t+1}h_{t+1}}{1+r_{t+1}} \tag{18}$$
$$w_t h_t l_t = \frac{\gamma w_{t+1}h_{t+1}}{1+r_{t+1}} \tag{19}$$

Substituting equation (17) into equation (2), the purchase of annuities is given by

$$s_t = \frac{\beta q(1-\rho)}{1+\beta q}[(1+r_t)b_t + w_t h_t] \tag{20}$$

Finally, from equation (1), the bequest transferred to the next generation is given by

$$b_{t+1} = \rho[(1+r_t)b_t + w_t h_t] - \frac{w_{t+1}h_{t+1}}{1+r_{t+1}} \tag{21}$$

which is also derived from equations (11) and (15).

The technology in the production sector is specified by a Cobb-Douglas form,

$$y_t = Ak_t^{\alpha}[h_t(1-l_t)]^{1-\alpha}$$

where y_t, k_t, h_t stand for output, physical capital, and human capital, respectively. $0 < \alpha < 1$ is the income share of physical capital, and $A > 0$ is a productivity parameter.

Assuming the factor markets are perfectively competitive, the factor prices satisfy

$$w_t h_t(1-l_t) = (1-\alpha)y_t \tag{22}$$

$$(1+r_t)k_t = \alpha y_t \tag{23}$$

The physical capital market clears when

$$k_{t+1} = s_t + b_{t+1} \tag{24}$$

Assuming the annuity market is competitive, the rate of return on annuities is given by

$$R_t = \frac{1+r_t}{q} \tag{25}$$

Since the model is closed, the good market clearing condition,

$$y_t = c_{1t} + qc_{2t} + e_t + k_{t+1}$$

is given by Walras' law.

2.1 Equilibrium and dynamics

In this section we examine the law of motion of education time l_t and derive the equilibrium.

First, substituting equations (22) and (23) into equation (19), we have

$$k_{t+1} = \frac{\alpha l_t(1-l_{t+1})}{\gamma(1-l_t)}y_t \tag{26}$$

Equation (26) stands for the investment ratio which is consistent to the labor market condition because individuals choose optimally time allocation, i.e., labor supply.

Second, substituting equations (20) and (21) into equation (24), and using equations (22) and (23), we have

$$\frac{1-\alpha l_{t+1}}{\alpha(1-l_{t+1})}k_{t+1} = \frac{\rho+\beta q}{1+\beta q}[(1+r_t)b_t + w_t h_t] \tag{27}$$

Multiplying both side of equation (27) by $(1+r_{t+1})$, and using (21) and (23), we have

$$\frac{1-\alpha l_{t+1}}{1-l_{t+1}}y_{t+1} = \frac{\rho+\beta q}{\rho(1+\beta q)}[(1+r_{t+1})b_{t+1} + w_{t+1}h_{t+1}] \tag{28}$$

From equations (27) and (28), we have

$$k_{t+1} = \frac{\rho\alpha(1-\alpha l_t)(1-l_{t+1})}{(1-l_t)(1-\alpha l_{t+1})} y_t \tag{29}$$

Equation (29) stands for the investment ratio which is consistent to the physical capital market condition.

Finally, equations (26) and (29) regulate the law of motion of education time such as

$$l_{t+1} = \rho\gamma + \frac{1}{\alpha} - \frac{\rho\gamma}{\alpha l_t} \tag{30}$$

[Figure 1 is here]

The first-order difference equation (30) has two stationary solutions, $\rho\gamma$ and $1/\alpha$. Figure 1 shows the curve intersects upward with the 45 degree line at $\rho\gamma$. Therefore, we have a unique constant time that is consistent to a balanced growth path equilibrium such as

$$l_t = \rho\gamma \tag{31}$$

Substituting equation (31) into equation (26), the investment in physical capital is given by

$$k_{t+1} = \rho\alpha y_t \tag{32}$$

Equation (18) gives the education good,

$$e_t = \frac{\rho(1-\gamma)(1-\alpha)}{1-\rho\gamma} y_t \tag{33}$$

and equation (27) gives the household full income,

$$(1+r_t)b_t + w_t h_t = \frac{\rho(1+\beta q)(1-\rho\gamma\alpha)}{(\rho+\beta q)(1-\rho\gamma)} y_t \tag{34}$$

which determines the consumption allocation such as

$$c_{1t} = \frac{\rho}{\rho+\beta q} \frac{(1-\rho)(1-\rho\gamma\alpha)}{1-\rho\gamma} y_t \tag{35}$$

$$c_{2t} = \frac{\beta}{\rho+\beta q} \frac{(1-\rho)(1-\rho\gamma\alpha)}{1-\rho\gamma} y_t \tag{36}$$

The growth rate on a balanced growth path is given by equation (11),

$$g = \rho(1+r) \tag{37}$$

where r is constant and independent of the longevity q.

Finally, from equation (34), the bequest received is given by

$$(1+r_t)b_t = \zeta y_t$$

where

$$\zeta = \frac{\rho(1+\beta q)(1-\rho\gamma\alpha) - (\rho+\beta q)(1-\alpha)}{(\rho+\beta q)(1-\rho\gamma)} \tag{38}$$

If $\zeta \geq 0$, then the bequest-constraint is not binding and the decentralized economy would achieve the social optimum[4]. If $\zeta < 0$, then the constraint is binding and the equilibrium would be suboptimal. The following proposition summarizes the result in this section.

Proposition 1 *If the bequest-constraint is not binding, the decentralized economy achieves the socially optimal allocations. Population aging decreases both consumption when young and old with the share of total consumption $(c_{1t} + qc_{2t})/y_t$ kept constant. The optimal rate of economic growth is independent of population aging.*

3 Bequest-constrained economy

In this section we examine a bequest-constrained economy, which implies $b_t = 0$ in each period. The optimality conditions consist of equations (1) with $b_t = b_{t+1} = 0$, (2), (3), (4), (5), (7), and (8) for $c_{1t}, c_{2t+1}, s_t, e_t, l_t, h_{t+1}, \lambda_t$.

As in the previous section, the envelope theorem gives

$$\frac{dV_t}{dh_t} = \lambda_t w_t$$

with which equations (7) and (8) give

$$\lambda_t e_t = \rho(1-\gamma)\lambda_{t+1} w_{t+1} h_{t+1} \quad (39)$$

$$\lambda_t w_t h_t l_t = \rho\gamma\lambda_{t+1} w_{t+1} h_{t+1} \quad (40)$$

Substituting equations (4), (5), (39), and (40) into the lifetime budget constraint, we have

$$\lambda_t w_t h_t = 1 + \beta q + \rho\lambda_{t+1} w_{t+1} h_{t+1}$$

which regulates the law of motion of $\lambda_t w_t h_t$. Since $0 < \rho < 1$, we have a unique solution of

$$\lambda_t w_t h_t = \frac{1+\beta q}{1-\rho} \quad (41)$$

Substituting equation (41) into equation (40), we have

$$l_t = \rho\gamma \quad (42)$$

which implies the time allocation is optimal.

Substituting equation (41) into equations (1), (2), and (39), we have

$$c_{1t} = \frac{1-\rho}{1+\beta q} w_t h_t \quad (43)$$

$$c_{2t+1} = \frac{\beta q(1-\rho)}{1+\beta q} w_t h_t R_{t+1} \quad (44)$$

$$e_t = \rho(1-\gamma) w_t h_t \quad (45)$$

$$s_t = \frac{\beta q(1-\rho)}{1+\beta q} w_t h_t \quad (46)$$

[4] The socially optimal allocations are shown in Appendix.

Substituting equation (46) into the physical capital market clearing condition, $k_{t+1} = s_t$, and using equations (22) and (42), we have the investment in physical capital such as

$$k_{t+1} = \frac{\beta q}{1+\beta q} \frac{(1-\rho)(1-\alpha)}{1-\rho\gamma} y_t \tag{47}$$

which is increasing in longevity q.

Finally, equations (43), (44), and (45) give the resource allocations as follows.

$$c_{1t} = \frac{1}{1+\beta q} \frac{(1-\rho)(1-\alpha)}{1-\rho\gamma} y_t \tag{48}$$

$$c_{2t} = \frac{\alpha}{q} y_t \tag{49}$$

$$e_t = \frac{\rho(1-\gamma)(1-\alpha)}{1-\rho\gamma} y_t \tag{50}$$

Equations (48) and (49) imply consumption is decreasing in longevity q. Equation (50) coincides with equation (33), i.e., the resource allocation of education is optimal. Since the accumulation of human capital is optimal and that of physical capital is increasing in q, the growth rate is increasing in q. In sum, we have the following proposition.

Proposition 2 *In a bequest-constrained economy, the time and resource allocation of education are optimal. Population aging decreases both consumption when young and old, and increases the physical capital investment. Population aging increases the growth rate.*

Table 1 summarizes the equilibrium allocations when the bequest is operative (section 2) and when it is not operative (section 3).

[Table 1 is here]

4 Population aging

In this section we combine the results in the previous sections to trace the path of equilibrium in a process of population aging. The critical condition is whether the bequest-constraint is binding or not. Remember that

$$(1+r_t)b_t = \zeta y_t$$

where

$$\zeta = \frac{\rho(1+\beta q)(1-\rho\gamma\alpha) - (\rho+\beta q)(1-\alpha)}{(\rho+\beta q)(1-\rho\gamma)} \tag{51}$$

At a glance, we know consumption when young is inefficiently large in a bequest constrained economy because the condition $\zeta \leq 0$ is equivalent to

$$\left(\frac{c_{1t}}{y_t}\right)^* \leq \left(\frac{c_{1t}}{y_t}\right)^{BC} \tag{52}$$

where an asterisk and a superscript BC stand for the solutions in the social optimum and in a bequest constrained economy, respectively.

The interpretation is straightforward. The optimal bequests would be negative if individuals were allowed to reallocate consumption when young to consumption when old. Otherwise, consumption when young would be larger relative to the social optimum.

Focusing on the survival rate q in equation (51), a condition $\zeta \leq 0$ is equivalent to

$$\rho\alpha(1-\rho\gamma) - \beta q \varphi(\rho) \leq 0 \tag{53}$$

where

$$\varphi(\rho) = \gamma\alpha\rho^2 - \rho + 1 - \alpha \tag{54}$$

First, assume that $\varphi(\rho) \leq 0$. Then the left hand side of equation (53) is positive, which implies $\zeta > 0$, i.e., the decentralized economy is socially optimal. Otherwise, the bequest-constraint could be binding. Specifically, we have the following results.

Lemma 3 *There exists a unique $\hat{\rho} \in [1-\alpha, 1]$ such that $\varphi(\rho) > 0$ for any $\rho \in (0, \hat{\rho})$. $\hat{\rho}$ is increasing in γ.*

Proof. From equation (54), we know $\varphi(0) = 1-\alpha > 0$ and $\varphi(1) = -\alpha(1-\gamma) \leq 0$. Therefore, there exists a unique $\hat{\rho} \in (0,1)$ such that $\varphi(\hat{\rho}) = 0$ and $\varphi(\rho) > 0$ for any $\rho \in (0, \hat{\rho})$. Solving this, we have

$$\hat{\rho} = \frac{2(1-\alpha)}{1 + \sqrt{1 - 4\gamma\alpha(1-\alpha)}}$$

which implies that $\hat{\rho}$ is increasing in γ. For $\gamma \in [0,1]$, we have $\hat{\rho} \in [1-\alpha, 1]$. ■

Proposition 4 *Assume that $\rho \in (0, \hat{\rho})$. The bequest-constraint is binding if and only if $q \geq \hat{q}$ where*

$$\hat{q} = \frac{\rho\alpha(1-\rho\gamma)}{\beta\varphi(\rho)} \tag{55}$$

$\hat{q}$ is increasing in ρ and decreasing in γ.

Proof. Lemma 3 assures $\varphi(\rho) > 0$. From equation (53), $\zeta \leq 0$ is equivalent to $q \geq \hat{q}$. Differentiating $\hat{q}$ with respect to ρ, we have

$$\frac{\partial \hat{q}}{\partial \rho} = \frac{\alpha(1-\alpha)(\gamma\rho^2 - 2\gamma\rho + 1)}{\beta\varphi^2} > 0$$

for any $\rho \in (0, \hat{\rho})$. Finally, $\partial\hat{q}/\partial\gamma < 0$ because $\varphi(\rho)$ increases with γ. ■

[Figure 2 is here]

Figure 2 illustrates the locus of $\hat{q}$ in (ρ, q) plane. We assume $\alpha = 1/3$ and $\beta = 1/2$. The elasticity of time input in human capital formation γ is set to be 0, 1/3, and 1. The curve shifts rightward as γ increases from zero to one as shown in Proposition 4. The space is divided into two regions. In the region below the curve ($q < \hat{q}$), the allocations are optimal because the bequest-constraint is not binding. In the upper region ($q \geq \hat{q}$), the bequest constraint is binding.

Let us assume that $\rho = 0.4$. In a process of population aging, the economy moves upward on the vertical line $\rho = 0.4$. Figure 2 shows $\hat{q} = 0.8$ when $\gamma = 1/3$. If the adult mortality rate is relatively high ($q < 0.8$), then the bequest constraint is not binding. The growth rate is independent of the survival rate, which implies that population aging is neutral to economic growth. If the survival rate exceeds the critical level ($q > 0.8$), however, the economy enters the bequest-constrained region and population aging has a positive growth effect because of the overinvestment of physical capital. To see this clearly, let us transform equation (55) into two transparent formulas,

$$\begin{aligned}
\frac{\beta\hat{q}}{1+\beta\hat{q}} &= \frac{\rho\alpha(1-\rho\gamma)}{(1-\rho)(1-\alpha)} \\
\frac{\beta\hat{q}}{\rho+\beta\hat{q}} &= \frac{\alpha(1-\rho\gamma)}{(1-\rho)(1-\rho\gamma\alpha)}
\end{aligned}$$

Then, Table 1 shows that

$$\left(\frac{k_{t+1}}{y_t}\right)^* \leq \left(\frac{k_{t+1}}{y_t}\right)^{BC} \tag{56}$$

$$\left(\frac{c_{2t}}{y_t}\right)^* \geq \left(\frac{c_{2t}}{y_t}\right)^{BC} \tag{57}$$

for $q \geq \hat{q}$. The equalities hold if and only if $q = \hat{q}$. The investment in physical capital is inefficiently large in a bequest-constrained economy because individuals have an incentive to save for their future consumption. Consumption when old is inefficiently small because individuals are not allowed to reallocate consumption when young to consumption when old as mentioned above.

5 Concluding remarks

In this paper, we have shown that population aging may promote economic growth if adult mortality risk is fairly improved. The reason is that the accumulation of physical capital is inefficiently high relative to the social optimum, although the accumulation of human capital is optimal. However, a higher growth rate does not imply the welfare is maximized because the bequest-constraint is binding. If it is the case, we need some kind of public transfer from young to old generations, e.g., pay-as-you-go public pensions to recover the social optimum.

Appendix [Social optimum]

A social planner maximizes

$$\sum_{t=0}^{\infty} \rho(\ln c_{1t} + \beta q \ln c_{2t+1})$$

subject to the resource constraint,

$$y_t = c_{1t} + qc_{2t} + e_t + k_{t+1}$$

and the technologies of production and human capital formation,

$$\begin{aligned} y_t &= Ak_t^{\alpha}[h_t(1-l_t)]^{1-\alpha} \\ h_{t+1} &= Be_t^{1-\gamma}(h_t l_t)^{\gamma} \end{aligned}$$

Let us set up the Lagrangian,

$$\Phi = \sum_{t=0}^{\infty} \rho^t \{\ln c_{1t} + \beta q \ln c_{2t+1} + \lambda_t(y_t - c_{1t} - qc_{2t} - e_t - k_{t+1}) + \mu_t[Be_t^{1-\gamma}(h_t l_t)^{\gamma} - h_{t+1}]\}$$

The first order conditions require

$$\frac{1}{c_{1t}} - \lambda_t = 0 \tag{A1}$$

$$\frac{\beta q}{c_{2t+1}} - \rho\lambda_{t+1} q = 0 \tag{A2}$$

$$-\lambda_t + \mu_t \frac{\partial h_{t+1}}{\partial e_t} = 0 \tag{A3}$$

$$\lambda_t \frac{\partial y_t}{\partial l_t} + \mu_t \frac{\partial h_{t+1}}{\partial l_t} = 0 \tag{A4}$$

$$-\lambda_t + \rho\lambda_{t+1} \frac{\partial y_{t+1}}{\partial k_{t+1}} = 0 \tag{A5}$$

$$-\mu_t + \rho\lambda_{t+1} \frac{\partial y_{t+1}}{\partial h_{t+1}} + \rho\mu_{t+1} \frac{\partial h_{t+2}}{\partial h_{t+1}} = 0 \tag{A6}$$

and the transversality conditions are $\lim_{t\to\infty} \rho^t \lambda_t k_{t+1} = \lim_{t\to\infty} \rho^t \mu_t h_{t+1} = 0$.

From equations (A4), (2), and (3), we have

$$\frac{(1-\alpha)\lambda_t y_t}{1-l_t} = \frac{\gamma \mu_t h_{t+1}}{l_t} \tag{A7}$$

Multiplying both side of equation (A6) by h_{t+1}, and using equations (2) and (3),

$$\mu_t h_{t+1} = \rho(1-\alpha)\lambda_{t+1} y_{t+1} + \rho\gamma\mu_{t+1} h_{t+2} \tag{A8}$$

Substituting equation (A7) into equation (A8), we have

$$\frac{l_t}{1-l_t}\lambda_t y_t = \frac{\rho\gamma}{1-l_{t+1}}\lambda_{t+1} y_{t+1}$$

Since the economy is adjusted immediately to a balanced growth path (as shown below), the optimal time allocation is given by

$$l^* = \rho\gamma \tag{A9}$$

The time spent in education is higher if individuals are more patient (a larger ρ) and/or the role of education time in human capital formation is more important (a higher γ).

Multiplying both side of equation (A3) by e_t, and using equation (3),

$$\lambda_t e_t = (1-\gamma)\mu_t h_{t+1} \tag{A10}$$

Substituting equations (A7) and (A9) into equation (A10), the resources devoted to education is given by

$$\frac{e_t}{y_t} = \frac{\rho(1-\gamma)(1-\alpha)}{1-\rho\gamma} \equiv \phi \tag{A11}$$

The allocation of education good is higher if individuals are more patient (a higher ρ), the income share of human capital is higher (a lower α), and/or the role of education good in human capital formation is more important (a lower γ).

Multiplying both side of equation (A5) by k_{t+1}, and using equation (2),

$$\lambda_t k_{t+1} = \rho\alpha\lambda_{t+1} y_{t+1} \tag{A12}$$

Substituting equations (A1), (A2), (A11), and (A12) into equation (1), the law of motion of $\lambda_t y_t$ is given by

$$(1-\phi)\lambda_t y_t = 1 + \frac{\beta q}{\rho} + \rho\alpha\lambda_{t+1} y_{t+1}$$

From equation (A11), we know $\rho\alpha/(1-\phi) < 1$. With the transversality conditions, we have

$$\lambda_t y_t = \frac{1}{1-\phi-\rho\alpha}\left(1+\frac{\beta q}{\rho}\right)$$

Substituting this into equations (A1) and (A2), the consumption allocations are given by

$$c_{1t}^* = \frac{\rho}{\rho+\beta q}\frac{(1-\rho)(1-\rho\gamma\alpha)}{1-\rho\gamma} y_t \tag{A13}$$

$$c_{2t}^* = \frac{\beta}{\rho+\beta q}\frac{(1-\rho)(1-\rho\gamma\alpha)}{1-\rho\gamma} y_t \tag{A14}$$

Finally, from equation (A12), the investment in physical capital is given by

$$k_{t+1}^* = \rho\alpha y_t \tag{A15}$$

Next, we derive the growth rate on the balanced growth path. First, substituting equation (A11) into equation (3), the growth rate of human capital is given by

$$g_H = \frac{h_{t+1}}{h_t} = B\phi^{1-\gamma}(l^*)^\gamma \left(\frac{y_t}{h_t}\right)^{1-\gamma} \tag{A16}$$

From equation (A15), the growth rate of physical capital is given by

$$g_K = \frac{k_{t+1}}{k_t} = \rho\alpha\frac{y_t}{k_t} \tag{A17}$$

From equation (2), the output-capital ratios are given by

$$\begin{aligned} \frac{y_t}{h_t} &= A(1-l^*)^{1-\alpha}\kappa_t^{\alpha} \\ \frac{y_t}{k_t} &= A(1-l^*)^{1-\alpha}\kappa_t^{\alpha-1} \end{aligned}$$

where κ stands for the capital ratio,

$$\kappa_t = \frac{k_t}{h_t}$$

Substituting them into equations (A16) and (A17), we know

$$\begin{aligned} g_H &= B(l^*)^{\gamma}\left[A\phi(1-l^*)^{1-\alpha}\kappa_t^{\alpha}\right]^{1-\gamma} && \text{(A18)} \\ g_K &= \rho\alpha A(1-l^*)^{1-\alpha}\kappa_t^{\alpha-1} && \text{(A19)} \end{aligned}$$

The condition for a balanced growth requires $g_H = g_K$. From equations (A18) and (A19), the capital ratio is given by

$$\kappa^* = \left[\frac{\rho\alpha A^{\gamma}(1-l^*)^{\gamma(1-\alpha)}}{B(l^*)^{\gamma}\phi^{1-\gamma}}\right]^{\frac{1}{1-\gamma\alpha}} \tag{A20}$$

Substituting equation (A20) into equation (A18) or (A19), the optimal growth rate is given by

$$g^* = \left\{A^{1-\gamma}B^{1-\alpha}(\rho\alpha)^{(1-\gamma)\alpha}(l^*)^{\gamma(1-\alpha)}[\phi(1-l^*)]^{(1-\gamma)(1-\alpha)}\right\}^{\frac{1}{1-\gamma\alpha}}$$

Finally, substituting equations (A9) and (A11) into them, we have

$$\kappa^* = (1-\rho\gamma)\left[\frac{A^{\gamma}\alpha(1-\alpha)^{-(1-\gamma)}}{B\gamma^{\gamma}(1-\gamma)^{1-\gamma}}\right]^{\frac{1}{1-\gamma\alpha}}$$

and

$$g^* = \rho\left[A\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right]^{\frac{1-\gamma}{1-\gamma\alpha}}\left[B\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{\frac{1-\alpha}{1-\gamma\alpha}} \tag{A21}$$

The growth rate consists of three factors. First, it is higher if individuals are more patient (a higher ρ). Second, the left square bracket stands for the growth factor of the production sector. Finally, the right square bracket stands for the growth factor of the education sector. Equation (A21) confirms that population aging is neutral to the optimal rate of economic growth.

At the end of appendix, we show this model does not have the transition process. The dynamics is characterized by a pair of first-order difference equations of $\lambda_t y_t$ and $\lambda_t y_t l_t/(1-l_t)$. Let us denote $a_t = \lambda_t y_t$ and $b_t = \lambda_t y_t l_t/(1-l_t)$. We have

$$\begin{aligned} b_t &= \rho\gamma a_t + \rho\gamma b_{t+1} \\ a_t &= 1 + \frac{\beta q}{\rho} + \frac{(1-\gamma)(1-\alpha)}{\gamma}b_t + \rho\alpha a_{t+1} \end{aligned}$$

which has a unique stationary solution of

$$\begin{aligned} a^* &= \frac{(1-\rho\gamma)(\rho+\beta q)}{\rho(1-\rho)(1-\rho\gamma\alpha)} \\ b^* &= \frac{\rho\gamma}{1-\rho\gamma} a^* \end{aligned}$$

Defining distance sequences to the stationary state by $\tilde{a}_t = a_t - a^*$ and $\tilde{b}_t = b_t - b^*$, we have

$$\begin{pmatrix} \tilde{a}_{t+1} \\ \tilde{b}_{t+1} \end{pmatrix} = \Omega \begin{pmatrix} \tilde{a}_t \\ \tilde{b}_t \end{pmatrix}$$

where

$$\Omega = \begin{pmatrix} \frac{1}{\rho\alpha} & -\frac{(1-\gamma)(1-\alpha)}{\rho\gamma\alpha} \\ -\frac{1}{\rho\alpha} & \frac{\alpha+(1-\gamma)(1-\alpha)}{\rho\gamma\alpha} \end{pmatrix}$$

Since the characteristic roots are $\rho^{-1} > 1$ and $(\rho\gamma\alpha)^{-1} > 1$, we need $\tilde{a}_t = \tilde{b}_t = 0$ for all t to guarantee the transversality conditions.

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Table 1 Social optimum versus Bequest constrained economy

	Social optimal	Bequest constrained
l	$\rho\gamma$	$\rho\gamma$
e_t/y_t	$\frac{\rho(1-\gamma)(1-\alpha)}{1-\rho\gamma}$	$\frac{\rho(1-\gamma)(1-\alpha)}{1-\rho\gamma}$
k_{t+1}/y_t	$\rho\alpha$	$\frac{\beta q}{1+\beta q}\frac{(1-\rho)(1-\alpha)}{1-\rho\gamma}$
c_{1t}/y_t	$\frac{\rho}{\rho+\beta q}\frac{(1-\rho)(1-\rho\gamma\alpha)}{1-\rho\gamma}$	$\frac{1}{1+\beta q}\frac{(1-\rho)(1-\alpha)}{1-\rho\gamma}$
c_{2t}/y_t	$\frac{\beta}{\rho+\beta q}\frac{(1-\rho)(1-\rho\gamma\alpha)}{1-\rho\gamma}$	$\frac{\alpha}{q}$

Figure 1: Dynamics of education time

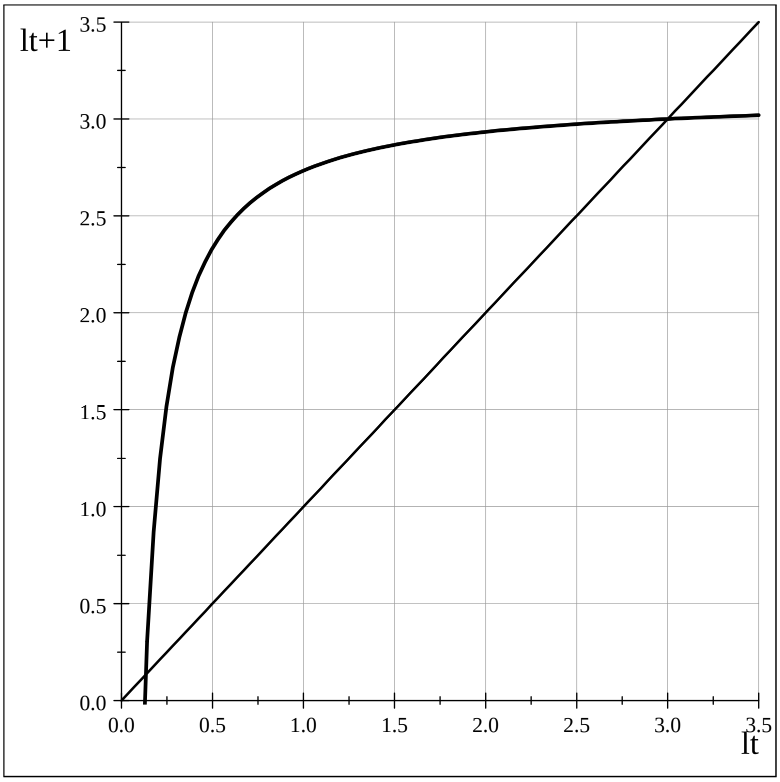

Figure 2: The locus of $\hat{q}$

Bequests are (not) operative in the region $q < \hat{q}$ ($q > \hat{q}$)